Math 201 Solution of Midterm Exam 2 for section 04 & 06

Q.1: Identify the quadratic surface $4x^2+6y^2-9z^2-8x+24y+18z=17$. Write what are (names only) its horizontal and vertical traces. **(10 pts)**

Sol: $4x^2-8x+6y^2+24y-9z^2+18z=17$

$4\left(x^2-2x\right)+6\left(y^2+4y\right)-9\left(z^2-2z\right)=17$

$4\left(x^2-2x+1\right)+6\left(y^2+4y+4\right)-9\left(z^2-2z+1\right)=17+4+24-9$

$4\left(x-1\right)^2+6\left(y+2\right)^2-9\left(z-1\right)^2=36$

$\dfrac{(x-1)^2}{9}+\dfrac{(y+2)^2}{6}-\dfrac{(z-1)^2}{4}=1$

The surface is hyperboild of one sheet.

Horizontal traces correspond to $z=k$ (constant) are ellipses.

Vertical traces correspond to $x=k$ or $y=k$ are hyperbolas.

Q.2: Two surfaces are tangent to each other at a point if they have the same tangent plane at that point. Show that the two surfaces $3x^2+2y^2+z^2=9$ and $x^2+y^2+z^2-8x-6y-8z+24=0$ are tangent to each other at $(1,1,2)$. **(10 pts)**

Sol: Let $F(x,y,z)=3x^2+2y^2+z^2-9$, and $G(x,y,z)=x^2+y^2+z^2-8x-6y-8z+24$

Then $\nabla F=\langle 6x,4y,2z\rangle$ and $\nabla G=\langle 2x-8,2y-6,2z-8\rangle$.

At $(1,1,2)$, $\nabla F(1,1,2)=\langle 6,4,4\rangle$ and $\nabla G(1,1,2)=\langle -6,-4,-4\rangle$.

Tangent plane to the surface $3x^2+2y^2+z^2=9$ at $(1,1,2)$ is $6(x-1)+4(y-1)+4(z-2)=0 \Rightarrow 6x+4y+4z=18$.

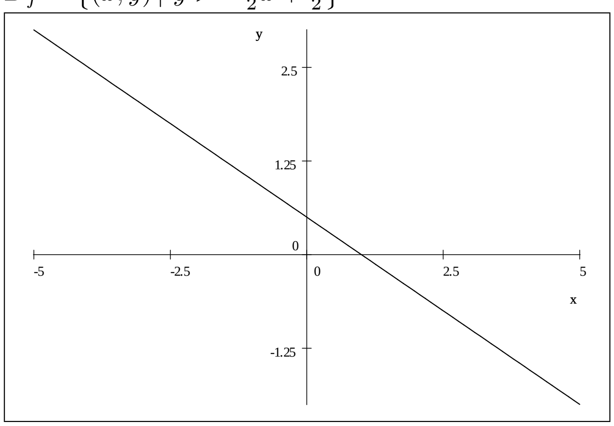
Tangent plane to the surface $x^2+y^2+z^2-8x-6y-8z+24=0$ at $(1,1,2)$ is $-6(x-1)-4(y-1)-4(z-2)=0 \Rightarrow 6x+4y+4z=18$.

Since the two tangent planes are same, the two surfaces are tangent to eachother.

Q.3: Let $f(x,y)=\log_{10}(x+2y-1)$

Sol: (a) Write and sketch the domain of f

$D_f=\left\{(x,y)\mid y>-\frac{1}{2}x+\frac{1}{2}\right\}$



(b) Evaluate $f(1,50)$ and $f\left(\sqrt{10},\dfrac{1}{2}\right)$. **(4 pts)**

$f(1,50)=\log_{10}(1+100-1)=\log_{10}\left(10^2\right)=2$

$f\left(\sqrt{10},\dfrac{1}{2}\right)=\log_{10}\left(\sqrt{10}+2\left(\dfrac{1}{2}\right)-1\right)=\log_{10}\left(10^{\frac{1}{2}}\right)=\dfrac{1}{2}.$

(c) Write the range of f. **(2 pts)**

$(-\infty,\infty)$.

Q.4: (a) Write the cartesian equation for the surface whose spherical equation is $\rho = \sin\phi\cos\theta$. **(4 pts)**

Sol: (a) Multiply both sides by ρ we get, $\rho^2 = \rho\sin\phi\cos\theta \Rightarrow x^2+y^2+z^2 = x \Rightarrow x^2-x+y^2+z^2=0$
$\Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 + z^2 = \dfrac{1}{4}$.

(b) Write the spherical equation for the surface whose cartesian equation is $x^2-y^2-2z^2=4$. **(3 pts)**
Using $x = \rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$,
we get $\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta - 2\rho^2\cos^2\phi = 4 \Rightarrow \rho^2\sin^2\phi\left(\cos^2\theta-\sin^2\theta\right) - 2\rho^2\cos^2\phi = 4$
$\Rightarrow \rho^2\left(\sin^2\phi\cos 2\theta - 2\cos^2\phi\right) = 4$.

(c) Find the limit using polar coordinates $\lim\limits_{(x,y)\to(0,0)} \cos^{-1}\left(\dfrac{x^3-xy^2}{x^2+y^2}\right)$. **(3 pts)**
Using $x = r\cos\theta$ and $y = r\sin\theta$, we get $\dfrac{x^3-xy^2}{x^2+y^2} = \dfrac{r^3\cos^3\theta - r^3\cos\theta\sin^2\theta}{r^2}$.
Also $r\to 0$ as $(x,y)\to(0,0)$.

$$\lim_{(x,y)\to(0,0)} \cos^{-1}\left(\frac{x^3-xy^2}{x^2+y^2}\right) = \lim_{r\to 0}\cos^{-1}\left(\frac{r^3\left(\cos^3\theta-\cos\theta\sin^2\theta\right)}{r^2}\right) = \lim_{r\to0}\cos^{-1}(r) = \cos^{-1}(0) = \frac{\pi}{2}.$$

Q.5: Explain why $f(x,y) = 3\sin x + 4\cos y$ is differentiable at the point $P\left(0,\dfrac{\pi}{2}\right)$. Also find the linearization $L(x,y)$ of $f(x,y)$ at the point P. **(10 pts)**

Sol: $f_x = 3\cos x$ and $f_y = -4\sin y$. At $P\left(0,\frac{\pi}{2}\right)$, $f_x\left(0,\frac{\pi}{2}\right) = 3$ and $f_y\left(0,\frac{\pi}{2}\right) = -4$. Both these derivatives are continuous which implies f is differentiable.
Linearization of f at $P\left(0,\frac{\pi}{2}\right)$ is:

$$L(x,y) = f\left(0,\frac{\pi}{2}\right) + f_x\left(0,\frac{\pi}{2}\right)(x-0) + f_y\left(0,\frac{\pi}{2}\right)\left(y-\frac{\pi}{2}\right) = 3x-4y+2\pi.$$

Q.6: Find the partial derivatives $f_{xx},\ f_{yy},\ f_{xy}$ for the function $f(x,y) = \tan(3x+2y)$. **(10 pts)**

Sol: $f_x = 3\sec^2(3x+2y)$, $f_{xx} = 3.2.3\sec(3x+2y)\sec(3x+2y)\tan(3x+2y) = 18\sec^2(3x+2y)\tan(3x+2y)$
$f_y = 2\sec^2(3x+2y)$, $f_{xx} = 2.2.2\sec(3x+2y)\sec(3x+2y)\tan(3x+2y) = 8\sec^2(3x+2y)\tan(3x+2y)$
$f_x = 3\sec^2(3x+2y)$, $f_{xy} = 3.2.2\sec(3x+2y)\sec(3x+2y)\tan(3x+2y) = 12\sec^2(3x+2y)\tan(3x+2y)$

Q.7: Use chain rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if $z = \tan^{-1}\left(x^2y^3\right)$, $x = \ln\left(s^3t\right)$, $y = \sec(s+t)$. **(10 pts)**

Sol:
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2xy^3}{1+x^4y^6}\frac{3s^2}{s^3t} + \frac{3x^2y^2}{1+x^4y^6}\sec(s+t)\tan(s+t)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2xy^3}{1+x^4y^6}\frac{1}{s^3t} + \frac{3x^2y^2}{1+x^4y^6}\sec(s+t)\tan(s+t).$$

Q.8: The length l, width w and height h of a box are changing with time. If l is increasing at a rate of $3\ m/s$, w is decreasing at a rate of $2\ m/s$, and h is increasing at a rate of $1\ m/s$. Find the rate of change of the volume of the box when $l = 10$, $w = 8$, and $h = 5$. **(10 pts)**

Sol: $\dfrac{dl}{dt} = 3$, $\dfrac{dw}{dt} = -2$, and $\dfrac{dh}{dt} = 1$. For $V = lwh$, we need to find $\dfrac{dV}{dt}$ when $l = 10$, $w = 8$, and $h = 5$.
$\dfrac{dV}{dt} = \dfrac{\partial V}{\partial l}\dfrac{dl}{dt} + \dfrac{\partial V}{\partial w}\dfrac{dw}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = wh\dfrac{dl}{dt} + lh\dfrac{dw}{dt} + lw\dfrac{dh}{dt} = (8)(5)(3) + (10)(5)(-2) + (10)(8)(1) =$
$100\ m^3/s$.

Q.9: Find the directional derivative of the function $f(x,y,z) = \dfrac{1}{2x+3y-z}$ at the point $(1,2,3)$ in the direction $v = \langle 1,2,3 \rangle$. **(10 pts)**

Sol: $|v| = \sqrt{1+4+9} = \sqrt{14}$ and $u = \left\langle \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right\rangle$.

$$f_x = \frac{-2}{(2x+3y-z)^2},\ f_y = \frac{-3}{(2x+3y-z)^2},\ f_z = \frac{1}{(2x+3y-z)^2}.$$

At $(1,2,3)$, $f_x(1,2,3) = \dfrac{-2}{25}$, $f_y(1,2,3) = \dfrac{-3}{25}$, $f_z(1,2,3) = \dfrac{1}{25}$.

$$D_u f(1,2,3) = \frac{-2}{25\sqrt{14}} + \frac{-6}{25\sqrt{14}} + \frac{3}{25\sqrt{14}} = \frac{-5}{25\sqrt{14}} = \frac{-1}{5\sqrt{14}}.$$

Q.10: If $u = f(x,y)$, where $x = e^s \cos t$ and $y = e^s \sin t$. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right]$. **(10 pts)**

Sol: First note that $\dfrac{\partial x}{\partial s} = e^s \cos t = x$, $\dfrac{\partial x}{\partial t} = -e^s \sin t = -y$, $\dfrac{\partial y}{\partial s} = e^s \sin t = y$, $\dfrac{\partial y}{\partial t} = e^s \cos t = x$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}e^s \cos t + \frac{\partial u}{\partial y}e^s \sin t = \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y.$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y\right)\frac{\partial y}{\partial s}$$

$$= \left(\frac{\partial^2 u}{\partial x^2}x + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}y\right)x + \left(\frac{\partial^2 u}{\partial x \partial y}x + \frac{\partial^2 u}{\partial y^2}y + \frac{\partial u}{\partial y}\right)y$$

$$= \frac{\partial^2 u}{\partial x^2}x^2 + \frac{\partial u}{\partial x}x + \frac{\partial^2 u}{\partial x \partial y}xy + \frac{\partial^2 u}{\partial x \partial y}xy + \frac{\partial^2 u}{\partial y^2}y^2 + \frac{\partial u}{\partial y}y$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x}(-e^s \sin t) + \frac{\partial u}{\partial y}e^s \cos t = \frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y}x$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y}x\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y}x\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y}x\right)\frac{\partial y}{\partial t}$$

$$= \left(\frac{\partial^2 u}{\partial x^2}(-y) + \frac{\partial^2 u}{\partial x \partial y}x + \frac{\partial u}{\partial y}\right)(-y) + \left(\frac{\partial^2 u}{\partial x \partial y}(-y) + \frac{\partial u}{\partial x}(-1) + \frac{\partial^2 u}{\partial y^2}x\right)x$$

$$= \frac{\partial^2 u}{\partial x^2}y^2 + \frac{\partial^2 u}{\partial x \partial y}(-xy) + \frac{\partial u}{\partial y}(-y) + \frac{\partial^2 u}{\partial x \partial y}(-xy) + \frac{\partial u}{\partial x}(-x) + \frac{\partial^2 u}{\partial y^2}x^2$$

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\left(x^2 + y^2\right) + \frac{\partial^2 u}{\partial y^2}\left(x^2 + y^2\right) = \left(x^2 + y^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = e^{2s}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right)$.