

Q.1: Identify the quadratic surface $4x^2 + 6y^2 - 9z^2 - 8x + 24y + 18z = 17$. Write what are (names only) its horizontal and vertical traces. **(10 pts)**

Sol: $4x^2 - 8x + 6y^2 + 24y - 9z^2 + 18z = 17$
 $4(x^2 - 2x) + 6(y^2 + 4y) - 9(z^2 - 2z) = 17$
 $4(x^2 - 2x + 1) + 6(y^2 + 4y + 4) - 9(z^2 - 2z + 1) = 17 + 4 + 24 - 9$
 $4(x - 1)^2 + 6(y + 2)^2 - 9(z - 1)^2 = 36$
 $\frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{6} - \frac{(z - 1)^2}{4} = 1$

The surface is hyperboloid of one sheet.

Horizontal traces correspond to $z = k$ (constant) are ellipses.

Vertical traces correspond to $x = k$ or $y = k$ are hyperbolas.

Q.2: Two surfaces are tangent to each other at a point if they have the same tangent plane at that point. Show that the two surfaces $3x^2 + 2y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at $(1, 1, 2)$. **(10 pts)**

Sol: Let $F(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$, and $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$

Then $\nabla F = \langle 6x, 4y, 2z \rangle$ and $\nabla G = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$.

At $(1, 1, 2)$, $\nabla F(1, 1, 2) = \langle 6, 4, 4 \rangle$ and $\nabla G(1, 1, 2) = \langle -6, -4, -4 \rangle$.

Tangent plane to the surface $3x^2 + 2y^2 + z^2 = 9$ at $(1, 1, 2)$ is $6(x - 1) + 4(y - 1) + 4(z - 2) = 0 \Rightarrow 6x + 4y + 4z = 18$.

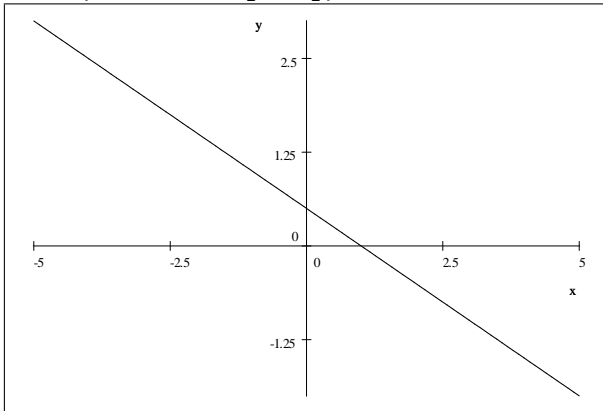
Tangent plane to the surface $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ at $(1, 1, 2)$ is $-6(x - 1) - 4(y - 1) - 4(z - 2) = 0 \Rightarrow 6x + 4y + 4z = 18$.

Since the two tangent planes are same, the two surfaces are tangent to each other.

Q.3: Let $f(x, y) = \log_{10}(x + 2y - 1)$

Sol: (a) Write and sketch the domain of f

$D_f = \{(x, y) \mid y > -\frac{1}{2}x + \frac{1}{2}\}$



(b) Evaluate $f(1, 50)$ and $f\left(\sqrt{10}, \frac{1}{2}\right)$. **(4 pts)**

$f(1, 50) = \log_{10}(1 + 100 - 1) = \log_{10}(10^2) = 2$

$f\left(\sqrt{10}, \frac{1}{2}\right) = \log_{10}\left(\sqrt{10} + 2\left(\frac{1}{2}\right) - 1\right) = \log_{10}\left(10^{\frac{1}{2}}\right) = \frac{1}{2}$.

(c) Write the range of f . **(2 pts)**

$(-\infty, \infty)$.

Q.4: (a) Write the cartesian equation for the surface whose spherical equation is $\rho = \sin \phi \cos \theta$. (4 pts)

Sol: (a) Multiply both sides by ρ we get, $\rho^2 = \rho \sin \phi \cos \theta \Rightarrow x^2 + y^2 + z^2 = x \Rightarrow x^2 - x + y^2 + z^2 = 0$
 $\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4}$.

(b) Write the spherical equation for the surface whose cartesian equation is $x^2 - y^2 - 2z^2 = 4$. (3 pts)

Using $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$,
we get $\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta - 2\rho^2 \cos^2 \phi = 4 \Rightarrow \rho^2 \sin^2 \phi (\cos^2 \theta - \sin^2 \theta) - 2\rho^2 \cos^2 \phi = 4$
 $\Rightarrow \rho^2 (\sin^2 \phi \cos 2\theta - 2 \cos^2 \phi) = 4$.

(c) Find the limit using polar coordinates $\lim_{(x,y) \rightarrow (0,0)} \cos^{-1} \left(\frac{x^3 - xy^2}{x^2 + y^2} \right)$. (3 pts)

Using $x = r \cos \theta$ and $y = r \sin \theta$, we get $\frac{x^3 - xy^2}{x^2 + y^2} = \frac{r^3 \cos^3 \theta - r^3 \cos \theta \sin^2 \theta}{r^2}$.

Also $r \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$.

$\lim_{(x,y) \rightarrow (0,0)} \cos^{-1} \left(\frac{x^3 - xy^2}{x^2 + y^2} \right) = \lim_{r \rightarrow 0} \cos^{-1} \left(\frac{r^3 (\cos^3 \theta - \cos \theta \sin^2 \theta)}{r^2} \right) = \lim_{r \rightarrow 0} \cos^{-1} (r) = \cos^{-1} (0) = \frac{\pi}{2}$.

Q.5: Explain why $f(x, y) = 3 \sin x + 4 \cos y$ is differentiable at the point $P \left(0, \frac{\pi}{2} \right)$. Also find the linearization $L(x, y)$ of $f(x, y)$ at the point P . (10 pts)

Sol: $f_x = 3 \cos x$ and $f_y = -4 \sin y$. At $P \left(0, \frac{\pi}{2} \right)$, $f_x \left(0, \frac{\pi}{2} \right) = 3$ and $f_y \left(0, \frac{\pi}{2} \right) = -4$. Both these derivatives are continuous which implies f is differentiable.

Linearization of f at $P \left(0, \frac{\pi}{2} \right)$ is:

$$L(x, y) = f \left(0, \frac{\pi}{2} \right) + f_x \left(0, \frac{\pi}{2} \right) (x - 0) + f_y \left(0, \frac{\pi}{2} \right) \left(y - \frac{\pi}{2} \right) = 3x - 4y + 2\pi.$$

Q.6: Find the partial derivatives f_{xx} , f_{yy} , f_{xy} for the function $f(x, y) = \tan(3x + 2y)$. (10 pts)

Sol: $f_x = 3 \sec^2(3x + 2y)$, $f_{xx} = 3 \cdot 2 \cdot 3 \sec(3x + 2y) \sec(3x + 2y) \tan(3x + 2y) = 18 \sec^2(3x + 2y) \tan(3x + 2y)$

$f_y = 2 \sec^2(3x + 2y)$, $f_{yy} = 2 \cdot 2 \cdot 2 \sec(3x + 2y) \sec(3x + 2y) \tan(3x + 2y) = 8 \sec^2(3x + 2y) \tan(3x + 2y)$

$f_{xy} = 3 \cdot 2 \cdot 2 \sec(3x + 2y) \sec(3x + 2y) \tan(3x + 2y) = 12 \sec^2(3x + 2y) \tan(3x + 2y)$

Q.7: Use chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if $z = \tan^{-1}(x^2 y^3)$, $x = \ln(s^3 t)$, $y = \sec(s + t)$. (10 pts)

Sol: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{2xy^3}{1+x^4y^6} \frac{3s^2}{s^3t} + \frac{3x^2y^2}{1+x^4y^6} \sec(s+t) \tan(s+t)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{2xy^3}{1+x^4y^6} \frac{1}{s^3t} + \frac{3x^2y^2}{1+x^4y^6} \sec(s+t) \tan(s+t).$$

Q.8: The length l , width w and height h of a box are changing with time. If l is increasing at a rate of 3 m/s, w is decreasing at a rate of 2 m/s, and h is increasing at a rate of 1 m/s. Find the rate of change of the volume of the box when $l = 10$, $w = 8$, and $h = 5$. (10 pts)

Sol: $\frac{dl}{dt} = 3$, $\frac{dw}{dt} = -2$, and $\frac{dh}{dt} = 1$. For $V = lwh$, we need to find $\frac{dV}{dt}$ when $l = 10$, $w = 8$, and $h = 5$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial l} \frac{dl}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = wh \frac{dl}{dt} + lh \frac{dw}{dt} + lw \frac{dh}{dt} = (8)(5)(3) + (10)(5)(-2) + (10)(8)(1) = 100 \text{ m}^3/\text{s}.$$

Q.9: Find the directional derivative of the function $f(x, y, z) = \frac{1}{2x + 3y - z}$ at the point $(1, 2, 3)$ in the direction $v = \langle 1, 2, 3 \rangle$. (10 pts)

Sol: $|v| = \sqrt{1 + 4 + 9} = \sqrt{14}$ and $u = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$.

$$f_x = \frac{-2}{(2x + 3y - z)^2}, \quad f_y = \frac{-3}{(2x + 3y - z)^2}, \quad f_z = \frac{1}{(2x + 3y - z)^2}.$$

$$\text{At } (1, 2, 3), \quad f_x(1, 2, 3) = \frac{-2}{25}, \quad f_y(1, 2, 3) = \frac{-3}{25}, \quad f_z(1, 2, 3) = \frac{1}{25}.$$

$$D_u f(1, 2, 3) = \frac{-2}{25\sqrt{14}} + \frac{-6}{25\sqrt{14}} + \frac{3}{25\sqrt{14}} = \frac{-5}{25\sqrt{14}} = \frac{-1}{5\sqrt{14}}.$$

Q.10: If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right]$. (10 pts)

Sol: First note that $\frac{\partial x}{\partial s} = e^s \cos t = x$, $\frac{\partial x}{\partial t} = -e^s \sin t = -y$, $\frac{\partial y}{\partial s} = e^s \sin t = y$, $\frac{\partial y}{\partial t} = e^s \cos t = x$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t = \frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y.$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y \right) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y \right) \frac{\partial y}{\partial s}$$

$$= \left(\frac{\partial^2 u}{\partial x^2} x + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} y \right) x + \left(\frac{\partial^2 u}{\partial x \partial y} x + \frac{\partial^2 u}{\partial y^2} y + \frac{\partial u}{\partial y} \right) y$$

$$= \frac{\partial^2 u}{\partial x^2} x^2 + \frac{\partial u}{\partial x} x + \frac{\partial^2 u}{\partial x \partial y} xy + \frac{\partial^2 u}{\partial x \partial y} xy + \frac{\partial^2 u}{\partial y^2} y^2 + \frac{\partial u}{\partial y} y$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial u}{\partial x} (-e^s \sin t) + \frac{\partial u}{\partial y} e^s \cos t = \frac{\partial u}{\partial x} (-y) + \frac{\partial u}{\partial y} x$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} (-y) + \frac{\partial u}{\partial y} x \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} (-y) + \frac{\partial u}{\partial y} x \right) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} (-y) + \frac{\partial u}{\partial y} x \right) \frac{\partial y}{\partial t}$$

$$= \left(\frac{\partial^2 u}{\partial x^2} (-y) + \frac{\partial^2 u}{\partial x \partial y} x + \frac{\partial u}{\partial y} \right) (-y) + \left(\frac{\partial^2 u}{\partial x \partial y} (-y) + \frac{\partial u}{\partial x} (-1) + \frac{\partial^2 u}{\partial y^2} x \right) x$$

$$= \frac{\partial^2 u}{\partial x^2} y^2 + \frac{\partial^2 u}{\partial x \partial y} (-xy) + \frac{\partial u}{\partial y} (-y) + \frac{\partial^2 u}{\partial x \partial y} (-xy) + \frac{\partial u}{\partial x} (-x) + \frac{\partial^2 u}{\partial y^2} x^2$$

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} (x^2 + y^2) + \frac{\partial^2 u}{\partial y^2} (x^2 + y^2) = (x^2 + y^2) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = e^{2s} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\text{Thus } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right).$$