Math 201 Midterm Exam 1 Solution
Dr. Muhammad Yousuf

Q.1: Let parametric equations of a curve are: $x(t) = 2 + 3t^2$ and $y(t) = 3 + 2t^3$.

(a) Find $\dfrac{dx}{dt}$ (1 pts) $\qquad \dfrac{dx}{dt} = 6t$

(b) Find $\dfrac{dy}{dt}$ (1 pts) $\qquad \dfrac{dy}{dt} = 6t^2$

(c) Find $\dfrac{dy}{dx}$ (2 pts) $\qquad \dfrac{dy}{dx} = t$

(d) Find $\dfrac{d^2y}{dx^2}$ (3 pts) $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1}{6t}$

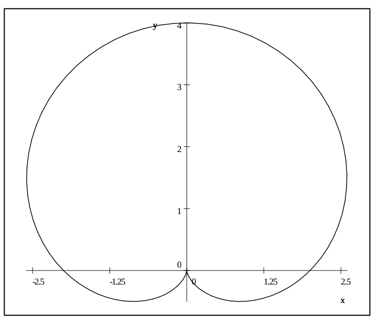
(e) Find length of the curve for $0 \le t \le 1$. (8 pts)

Arc Length $s = \int_0^1 \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} dt = \int_0^1 \sqrt{36t^2 + 36t^4} dt$

$= 3\int_0^1 2t\sqrt{1+t^2} dt = 3\dfrac{2}{3}\left(1+t^2\right)^{\frac{3}{2}}\Big|_0^1 = 2\left(2\sqrt{2} - 1\right)$

Q.2: Sketch graph of the polar curve $r = 2 + 2\sin(\theta)$. Show the points on the curve for $\theta = 0$, $\dfrac{\pi}{2}$, π, $\dfrac{3\pi}{2}$, 2π.(4 pts)

θ	r
0	2
$\frac{\pi}{2}$	4
π	2
$\frac{3\pi}{2}$	0
2π	2



$x = r\cos(\theta) = (2 + 2\sin(\theta))\cos(\theta) = 2\cos(\theta) + 2\sin(\theta)\cos(\theta) = 2\cos(\theta) + \sin(2\theta)$
$y = r\sin(\theta) = (2 + 2\sin(\theta))\sin(\theta) = 2\sin(\theta) + 2\sin^2(\theta)$

(a) Find $\dfrac{dx}{d\theta}$ (2 pts) $\qquad \dfrac{dx}{d\theta} = -2\sin(\theta) + 2\cos(2\theta)$

(b) Find $\dfrac{dy}{d\theta}$ (2 pts) $\qquad \dfrac{dy}{d\theta} = 2\cos(\theta) + 4\sin(\theta)\cos(\theta) = 2\cos(\theta) + 2\sin(2\theta)$

(c) Find $\dfrac{dy}{dx}$ (2 pts) $\qquad \dfrac{dy}{dx} = \dfrac{2\cos(\theta) + 2\sin(2\theta)}{-2\sin(\theta) + 2\cos(2\theta)}$

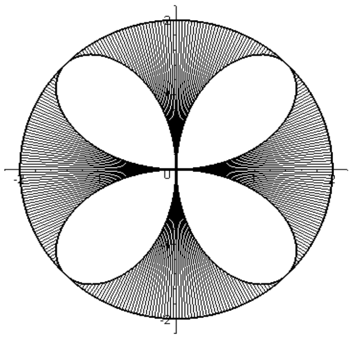
(d) Find equation of tangent line at $\theta = \dfrac{\pi}{4}$ (5 pts)

Slope $m = \dfrac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = \dfrac{2\cos(\theta) + 2\sin(2\theta)}{-2\sin(\theta) + 2\cos(2\theta)}\Big|_{\theta=\frac{\pi}{4}} = \dfrac{2\cos(\frac{\pi}{4}) + 2\sin(2\frac{\pi}{4})}{-2\sin(\frac{\pi}{4}) + 2\cos(2\frac{\pi}{4})} = -\sqrt{2} - 1$

Also at $\theta = \dfrac{\pi}{4}$, $x = 2\cos(\dfrac{\pi}{4}) + \sin(2\dfrac{\pi}{4}) = \sqrt{2} + 1$ and $y = 2\sin(\dfrac{\pi}{4}) + 2\sin^2(\dfrac{\pi}{4}) = \sqrt{2} + 1$

Equation of tangemt line is $\left(y - \sqrt{2} - 1\right) = \left(-\sqrt{2} - 1\right)\left(x - \sqrt{2} - 1\right)$.

Q.3: Find area of the region that lies inside $r = 2$ and outside $r = 2\sin(2\theta)$. (10 pts)



Area of one loop is $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin^2(t)dt = \frac{1}{2}\pi$. Total area of rose is $4A = 2\pi$ and area outside of rose and inside of circle is $4\pi - 2\pi = 2\pi$.

Q.4: If $r = \langle x, y, z\rangle$, $a = \langle 2, 3, 1\rangle$, and $b = \langle -2, 1, 5\rangle$. Show that the vector equation $(r-a)\cdot(r-b)$ represents a sphere. Find center and radius of the sphere (10 pts)

$(r-a) = \langle x-2, y-3, z-1\rangle$ and $(r-b) = \langle x+2, y-1, z-5\rangle$

$(r-a)\cdot(r-b) = (x-2)(x+2) + (y-3)(y-1) + (z-1)(z-5) = x^2 - 6z - 4y + y^2 + z^2 + 4$

$(r-a)\cdot(r-b) = 0 \Rightarrow x^2 - 6z - 4y + y^2 + z^2 + 4 = 0 \Rightarrow x^2 + y^2 - 4y + 4 + z^2 - 6z + 9 = 9 \Rightarrow x^2 + (y-2)^2 + (z-3)^2 = 3^2$

The equation represents sphere with center at $(0, 2, 3)$ and radius equal to 3.

Q.5: Find scalar and vector projection of the vector $\mathbf{u} = \mathbf{3}\ \mathbf{i} - \mathbf{j} + 2\ \mathbf{k}$ in the direction of $\mathbf{v} = \mathbf{2}\ \mathbf{i} + 3\ \mathbf{j} - 4\ \mathbf{k}$. (5+5 pts)

$Comp_{\mathbf{v}}\ \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{6-3-8}{\sqrt{29}} = \frac{-5}{\sqrt{29}}$ and $Proj_{\mathbf{v}}\ \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) = \frac{-5}{\sqrt{29}}\left(\frac{\mathbf{2}\ \mathbf{i} + 3\ \mathbf{j} - 4\ \mathbf{k}}{\sqrt{29}}\right) = \frac{-10\ \mathbf{i} - 15\ \mathbf{j} + 20\ \mathbf{k}}{29}$

Q.6: Determine whether the three vectors $\mathbf{u} = \mathbf{i} + 4\ \mathbf{j} - 7\ \mathbf{k}$, $\mathbf{v} = 2\ \mathbf{i} - \mathbf{j} + 4\ \mathbf{k}$, and $\mathbf{w} = 0\ \mathbf{i} - 9\ \mathbf{j} + 18\ \mathbf{k}$ are coplanar or not. Show all your work. (10 pts)

$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 18i - 36j - 18k$ and $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 18 + 4(-36) - 7(-18) = 0.$

The three vectors are coplanar.

Q.7: Find equation of a line containing the point $(1, 2, -3)$ and is perpendicular to the plane $3x - 2y + z = 5$. Write your answer in symmetric and parametric form. (10 pts)

Since line is perpendicular to the plane therefore it is parallel to the normal of the plane which is $\vec{n} = \langle 3, -2, 1\rangle$. Thus equation of the line is:

Symmetric form $\frac{(x-1)}{3} = \frac{(y-2)}{-2} = \frac{(z+3)}{1} = t$

Parametric form $x = 3t + 1,\ y = -2t + 2,\ z = t - 3.$

Q.8: Find equation of a plane passing through the points $(1, 2, 3)$, $(-3, 4, 2)$, and $(2, 4, 5)$. (10 pts)

Let $P(1, 2, 3)$, $Q(-3, 4, 2)$, and $R(2, 4, 5)$. Then two vectors in the plane are

$P\vec{Q} = \langle -4, 2, -1\rangle$ and $P\vec{R} = \langle 1, 2, 2\rangle$. Cross product of these two vectors is a vector normal to the plane.

$P\vec{Q}\times P\vec{R} = \begin{vmatrix} i & j & k \\ -4 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 6i + 7j - 10k$. Thus equation of the plane passing through $P(1, 2, 3)$ and $\vec{n} = 6i + 7j - 10k$ is $6x + 7y - 10z = 1(6) + 2(7) + 3(-10) = -10$.

Q.9: Find angle between the planes $3x - 2y + z = 5$ and $2x + y - 4z = 5$. (5 pts)

Normal to these planes are $\vec{n}_1 = 3i - 2j + k$ and $\vec{n}_2 = 2i + j - 4k$. Angle between two planes is the angle between their normals. Thus $\cos(\theta) = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

Q.10: Prove that $(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v}) = \mathbf{2}(\mathbf{u}\times\mathbf{v})$ (5 pts)

$(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = \mathbf{0} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} - \mathbf{0} = \mathbf{2}(\mathbf{u}\cdot\mathbf{v})$.