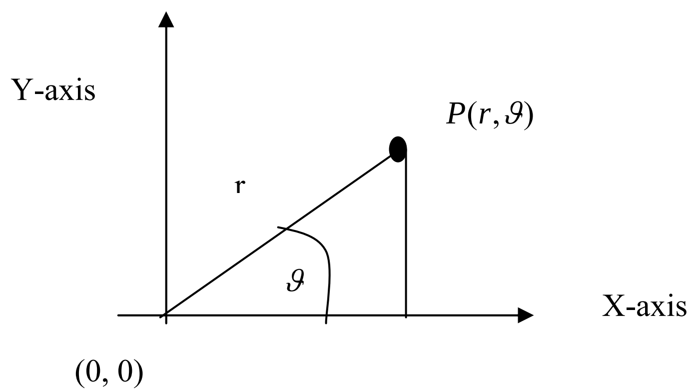
10.3 Polar Coordinates

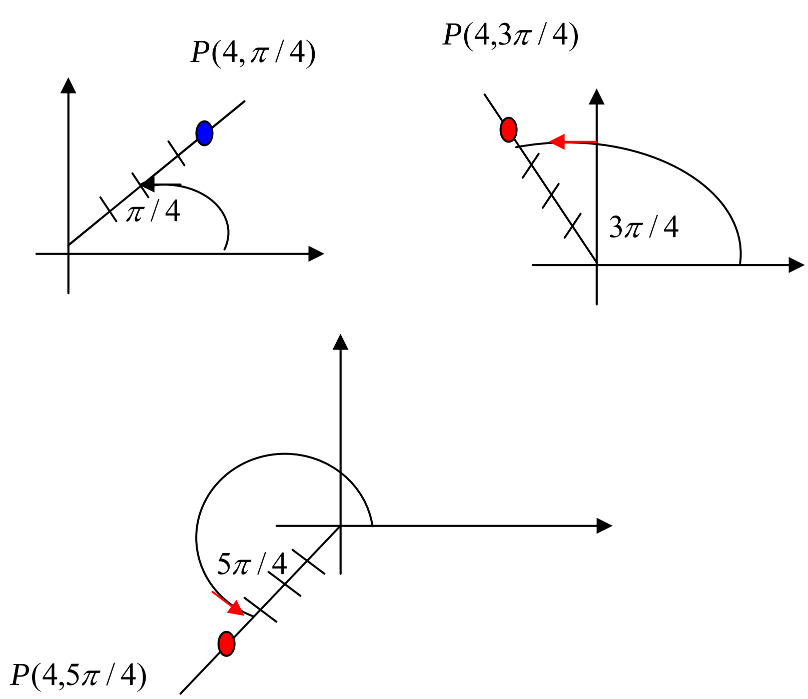
• Objectives and Outcomes

- We introduce polar coordinates
- We learn how to use these coordinates to draw curves. Such curves are called polar curves.
- We define tangents the polar curves.
- We apply these concepts to some questions from the exercise.

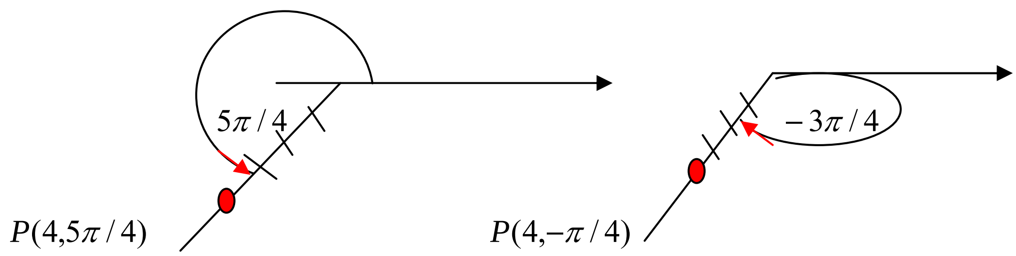
• Polar coordinates



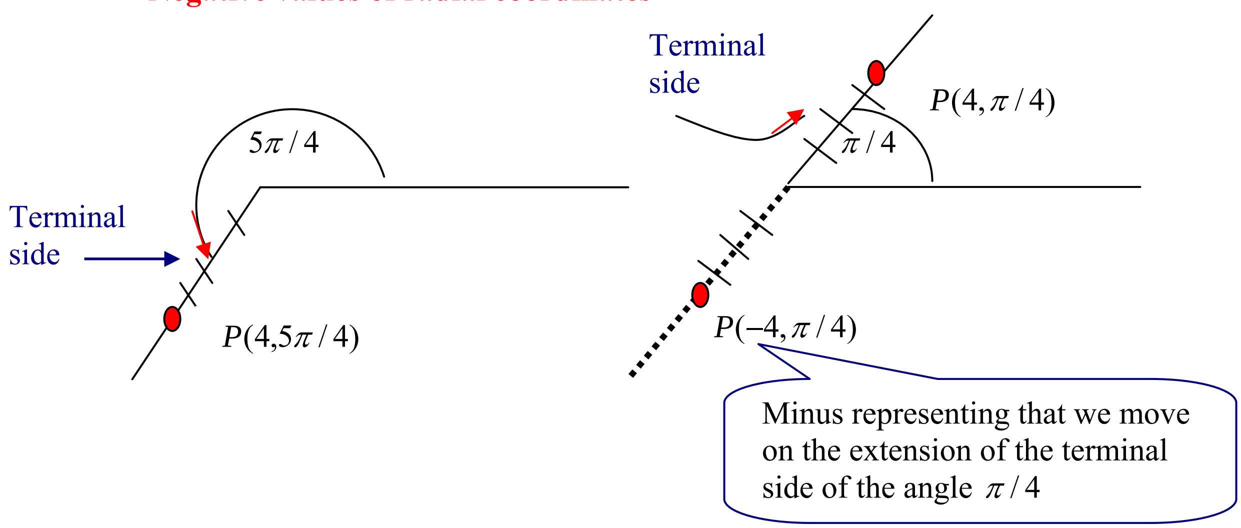
- General point in polar coordinates is represented by: (r,ϑ)
- r is called distance from origin to the point
- ϑ is the angle r makes with the polar axis (the x-axis)
- ϑ is positive in counter clock-wise and negative in clock-wise direction
- At pole, $r = 0$ and therefore pole is represented by: $(0,\vartheta)$
- Polar coordinates of some points can be drawn as follows:



- Polar coordinates of a point are: **Non-Unique**



➢ **Negative values of radial coordinates**



❖ Point $P(r, \vartheta)$ is the coordinate of the terminal side of the angle ϑ
❖ Point $P(-r, \vartheta)$ is the **extension** of the terminal side of the angle ϑ
❖ Thus $P(-r, \vartheta)$ and $P(r, \vartheta + \pi)$ are the coordinates of the same point.

- **Polar coordinates**
  - From first figure note that:

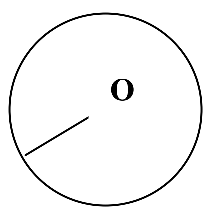
$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

$$\therefore \cos\vartheta = x/r, \quad \sin\vartheta = y/r$$

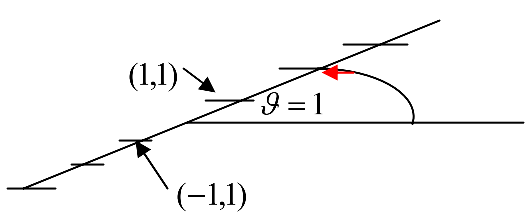
$$\therefore r = \sqrt{x^2 + y^2}, \quad \vartheta = \tan^{-1} y/x$$

- **Example 1** Convert the point $(2, \pi/3)$ into Cartesian coordinates.
  - $x = 2\cos\pi/3 = 1$
    $y = 2\sin\pi/3 = \sqrt{3}$
  - Thus Cartesian coordinates representing above point are: $(x, y) = (1, \sqrt{3})$

- **Example2** Represent the Cartesian point $(-2,2\sqrt{3})$ in terms of polar coordinates
  - The **Cartesian** point lies in the **second** quadrant
  - $r=\sqrt{x^2+y^2}=4$
    $\vartheta=\tan^{-1}(-\sqrt{3})=-\tan^{-1}\sqrt{3}=-\pi/3$
  - Thus the point lies on the extension of the terminal point of $-\pi/3$, i.e., at $\vartheta=-\pi/3+\pi=2\pi/3$
  - Thus coordinates of the polar point are: $(r,\vartheta)=(4,2\pi/3)$
  - All polar coordinates of the point are:
    $(4,2\pi/3+2n\pi)$ and $(-4,\{2\pi/3+\pi\}+2n\pi)$, with n integer.
- **Polar Graphs/Polar Curves**
  - A polar curve is defined by $r=f(\vartheta)$ or $F(r,\vartheta)=0$
- **Example3** Find curve that is represented by $r=2$
  - This curve consists of all points (r,ϑ) with $r=2$, which is a circle radius 2.

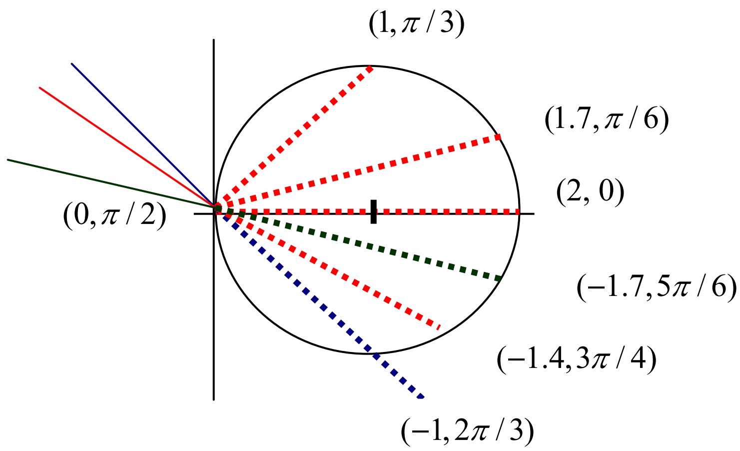


- **Example4** The equation $r=a$ represents a circle of radius $r=|a|$.
- **Example5** Plot the curve $\vartheta=1$.
  - The curve consists of all points (r,ϑ) such that $\vartheta=1$



- **Example6** Plot the curve $r = 2\cos\vartheta$
  - The curve is a set of point (r, ϑ) for $0 \le \vartheta \le 2\pi$.

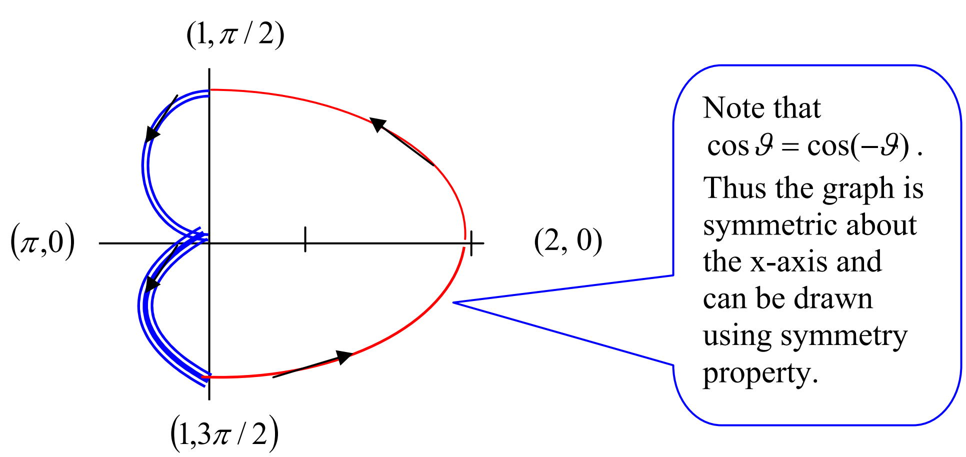
ϑ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	1.7	1	0	-1	-1.4	-1.7	-2



- **Example7** Convert polar coordinates $r = 2\cos\vartheta$ to Cartesian ones.
  - $x = r\cos\vartheta \Rightarrow x/r = \cos\vartheta$
  - Use it in $r = 2\cos\vartheta$ to get: $r = 2x/r \Rightarrow 2x = r^2$
  - Use $r^2 = x^2 + y^2$ above to get: $2x = x^2 + y^2$
  - Simplify last equation to get: $(x^2 - 1)^2 + (y - 0)^2 = 1$
  - Note that: Cartesian equation of circle tell you that the above curve is a circle of radius 1 with centre $(1,0)$

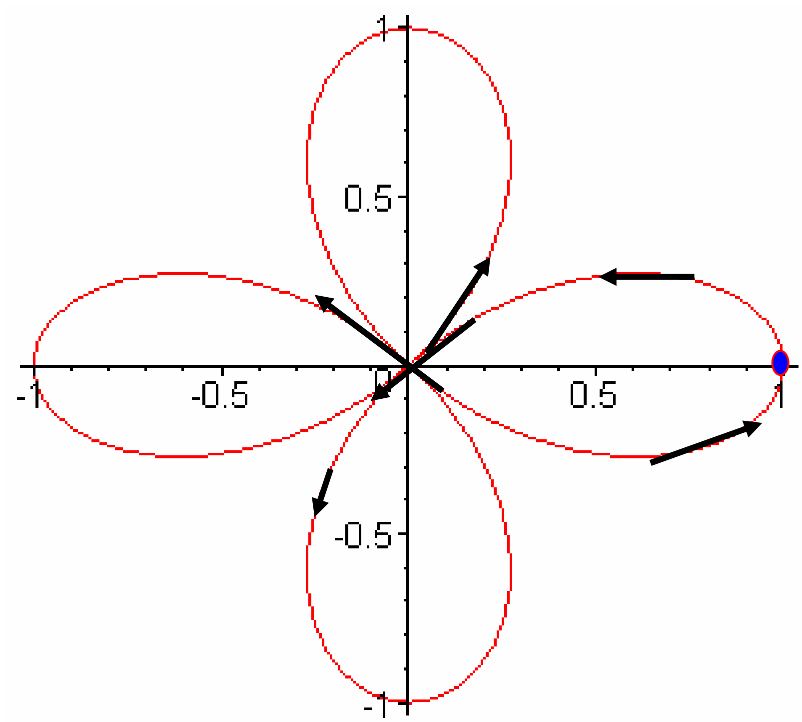
- **Example8** Draw the curve $r = 1 + \cos\vartheta$

ϑ	0	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	2	1.7	1.5	1	.29	0	.29	1	1.7	2



● **Example9** Draw the curve $r = \cos 2\vartheta$

2ϑ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π	$\frac{13\pi}{4}$	$\frac{7\pi}{2}$	$\frac{15\pi}{4}$	4π
ϑ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π	$\frac{9\pi}{8}$	$\frac{5\pi}{4}$	$\frac{11\pi}{8}$	$\frac{3\pi}{2}$	$\frac{13\pi}{8}$	$\frac{7\pi}{4}$	$\frac{15\pi}{8}$	2π
deg	0	22.5	45	67	90	112	135	157	180	198	225	242	270	286	315	337	360
r	**1**	**.7**	**0**	**-.7**	**-1**	**-.7**	**0**	**.7**	**1**	.7	0	-.7	-1	-.7	0	.7	1

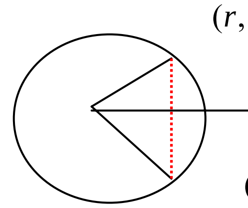


● Note on Symmetry of Polar Coordinates

- From above examples you would have realized that plotting is time consuming.
- It is therefore advisable to use symmetry of polar coordinates if it exists. The symmetry is defined as follows:

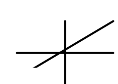
➢ **Symmetry about Polar Axis**

Polar Equation un-changed **<u>under</u>** $\vartheta \to -\vartheta$



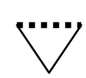
➢ **Symmetry about Pole:**

Polar Equation unchanged **<u>under</u>** $r \to -r$ or $\vartheta \to \vartheta + \pi$



➢ **Symmetry about Vertical line** $\vartheta = \pi/2$

Polar equation un-changed **<u>under</u>** $\vartheta \to \vartheta - \pi$ $(r,\vartheta-\pi)$



(r,ϑ)

● Tangents to Polar Curves

Consider a polar curve given by $r = f(\vartheta)$, then the parametric equations representing this curve can be given by,

$$x = f(\vartheta)\cos\vartheta, \quad y = f(\vartheta)\sin\vartheta$$

For the above curve we define tangent as:

- $dy/dx = \dfrac{dy/d\vartheta}{dx/d\vartheta} = \dfrac{f'\sin\vartheta + f\cos\vartheta}{f'\cos\vartheta - f\sin\vartheta}$
- **Suppose you are given ϑ at which the tangent is to be found, just plug that value in the above expression.**

Example 10 Determine equation of the tangent to the polar curve given by $r = 3 + 8\sin\vartheta$ at $\vartheta = \pi/6$.

- $dy/dx = \dfrac{dy/d\vartheta}{dx/d\vartheta} = \dfrac{16\cos\vartheta\sin\vartheta + 3\cos\vartheta}{8\cos^2\vartheta - 3\sin\vartheta - 8\sin^2\vartheta}$
- To find equation of the tangent line ‘ $y - y_1 = m(x - x_1)$ we now need “m” and (x, y)
- Slope ‘ $m = dy/dx\big|_{\vartheta=\pi/6}$ ’ of the tangent line at $\vartheta = \pi/6$ is: $\dfrac{11\sqrt{3}}{5}$
- At $\vartheta = \pi/6$, $r = 3 + 8\sin(\pi/6) = 7$, **therefore**
- $\boxed{x = r\cos\vartheta\big|_{(7,\pi/6)} = \dfrac{7\sqrt{3}}{2}}$ and $\boxed{x = r\sin\vartheta\big|_{(7,\pi/6)} = \dfrac{7}{2}}$
- The tangent line is: $y = \dfrac{7}{2} + \dfrac{11\sqrt{3}}{5}\left(x - \dfrac{7\sqrt{3}}{2}\right)$