Q:1 (a) (10 points) Evaluate the integral $\int_C x(x+y^2)dx + ydy$, along the curve C given by $x = \sqrt{2t}, y = t,\ 1 \le t \le 2.$

$$dx = \frac{1}{\sqrt{2t}}\,dt, \quad dy = dt$$

$$\int_C x(x+y^2)\,dx + y\,dy = \int_1^2 (\sqrt{2t} + t^2)\,dt + t\,dt$$

$$= \int_1^2 (t^2 + t + \sqrt{2t})\,dt$$

$$= \frac{t^3}{3} + \frac{t^2}{2} + \sqrt{2}\cdot\frac{2}{3}t^{\frac{3}{2}}\Big|_1^2$$

$$= \frac{8}{3} + 2 + \frac{8}{3} - \frac{1}{3} - \frac{1}{2} - \frac{2\sqrt{2}}{3}$$

$$= 5 + \frac{3}{2} - \frac{2\sqrt{2}}{3}$$

$$= \frac{13}{2} - \frac{2\sqrt{2}}{3}$$

Q:2 (a) (7 points) Find the directional derivative of $f(x,y,z) = 2xz + 3xy^2 + yz^2$ at $(-1,1,2)$ in the direction of $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

(b) (6 points) Write the direction of maximum directional derivative and value of maximum directional derivative.

(a) $\nabla f = \langle 2z + 3y^2,\ 6xy + z^2,\ 2x + 2yz \rangle$

$\nabla f(-1,1,2) = \langle 4+3,\ -6+4,\ -2+4 \rangle$

$= \langle 7, -2, 2 \rangle = 7\hat{i} - 2\hat{j} + 2\hat{k}$

$\|\vec{u}\| = \sqrt{4+9+36} = 7$

$\hat{u} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

$D_{\hat{u}} f(-1,1,2) = \nabla f \cdot \hat{u} = 2 - \frac{6}{7} + \frac{12}{7}$

$= \frac{20}{7}$

(b) Direction of maximum directional derivative

is $\nabla f(-1,1,2) = 7\hat{i} - 2\hat{j} + 2\hat{k}$

Value of maximum directional derivative

is $\|\nabla f(-1,1,2)\| = \sqrt{49+4+4} = \sqrt{57}$

Q:3 (13 points) Determine whether the vector field $\vec{F}(x,y,z) = (2x\sin y + e^{3z})\,\mathbf{i} + (x^2\cos y)\,\mathbf{j} + (3xe^{3z}+5)\,\mathbf{k}$ is a gradient field. If so, find the potential function $\phi(x,y,z)$ for $\vec{F}$.

$$P = 2x\sin y + e^{3z}, \quad Q = x^2\cos y, \quad R = 3xe^{3z} + 5$$

$$\frac{\partial P}{\partial y} = 2x\cos y, \quad \frac{\partial P}{\partial z} = 3e^{3z}, \quad \frac{\partial R}{\partial x} = 3e^{3z}$$

$$\frac{\partial Q}{\partial x} = 2x\cos y, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial y} = 0$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

Therefore $\vec{F}$ is a gradient field. There exists the potential function ϕ such that $d\phi = \vec{F}$.

$$\frac{\partial \phi}{\partial x} = P = 2x\sin y + e^{3z} \Rightarrow \phi(x,y,z) = x^2\sin y + xe^{3z} + g(y,z)$$

$$\frac{\partial \phi}{\partial y} = x^2\cos y + \frac{\partial g}{\partial y} = Q = x^2\cos y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$$

So $\phi = x^2\sin y + xe^{3z} + h(z)$

$$\frac{\partial \phi}{\partial z} = 3xe^{3z} + h'(z) = R = 3xe^{3z} + 5$$

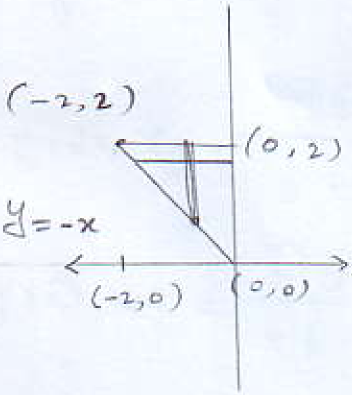
$$\Rightarrow h'(z) = 5 \Rightarrow h(z) = 5z$$

$$\phi(x,y,z) = x^2\sin y + xe^{3z} + 5z$$

Q:4 (12 points) Use Green's Theorem to evaluate the integral $\oint_C 3e^{-x^2}dx + 2\tan^{-1}x\,dy$, where C is the positively oriented triangle with vertices $(0,0)$, $(0,2)$, $(-2,2)$.

$P = 3e^{-x^2}$, $Q = 2\tan^{-1}x$

$\frac{\partial P}{\partial y} = 0$, $\frac{\partial Q}{\partial x} = \frac{2}{1+x^2}$



$$\oint_C 3e^{-x^2}dx + 2\tan^{-1}x\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{-2}^{0}\int_{-x}^{2} \frac{2}{1+x^2}\,dy\,dx = \int_{-2}^{0} \frac{2y}{1+x^2}\Big|_{-x}^{2} dx$$

$$= \int_{-2}^{0}\left(\frac{4}{1+x^2} + \frac{2x}{1+x^2}\right)dx = 4\tan^{-1}x + \ln(1+x^2)\Big|_{-2}^{0}$$

$$= 0 - 4\tan^{-1}(-2) - \ln 5 = 4\tan^{-1}2 - \ln 5$$

<u>OR</u>

$$\int_{0}^{2}\int_{-y}^{0} \frac{2}{1+x^2}\,dx\,dy = 2\int_{0}^{2} \tan^{-1}x\Big|_{-y}^{0} dy$$

$$= 2\int_{0}^{2} 1\cdot\tan^{-1}y\,dy$$

$$= 2\left[y\tan^{-1}y\Big|_{0}^{2} - \int_{0}^{2} y\cdot\frac{1}{1+y^2}\,dy\right]$$ Integration by parts

$$= 2\left[2\tan^{-1}2 - \frac{1}{2}\ln(1+y^2)\Big|_{0}^{2}\right]$$

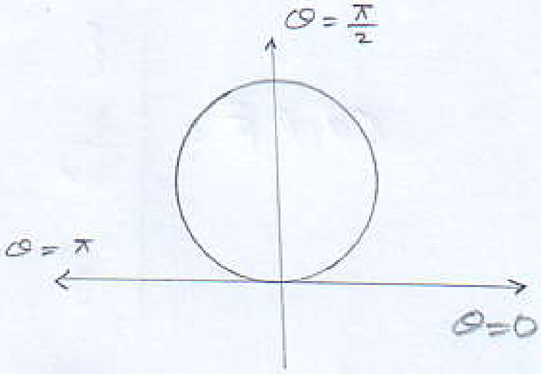
$$= 4\tan^{-1}2 - \ln 5$$

Q:5 (14 points) Find the surface area of the portions of the sphere $x^2 + y^2 + z^2 = 25$ that are within the cylinder $x^2 + y^2 = 5y$.

$$x^2 + y^2 = 5y$$

$$\Rightarrow \quad r^2 = 5r\sin\theta$$

$$\Rightarrow \quad r = 5\sin\theta$$



$$z = f(x,y) = \sqrt{25 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{25 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA$$

$$A(S) = 2\cdot 5 \iint_R \frac{1}{\sqrt{25 - (x^2 + y^2)}}\, dA$$

one upper and one lower portion

$$= 2\cdot 5\cdot 2 \int_0^{\frac{\pi}{2}} \int_0^{5\sin\theta} \frac{1}{\sqrt{25 - r^2}}\, r\, dr\, d\theta$$

$$= 20 \int_0^{\frac{\pi}{2}} \frac{-1}{2}\cdot\frac{2}{1}(25 - r^2)^{\frac{1}{2}} \Big|_0^{5\sin\theta}\, d\theta$$

$$= -20 \int_0^{\frac{\pi}{2}} (5\cos\theta - 5)\, d\theta$$

$$= 100\left[\theta - \sin\theta\right]_0^{\frac{\pi}{2}} = 100\left[\frac{\pi}{2} - 1\right]$$

$$= 50\pi - 100$$

Q;6 (14 points) Let $\vec{F}(x,y,z) = y^3\mathbf{i} - \mathbf{x}^3\mathbf{j} + z^3\mathbf{k}$. Use Stokes' theorem to evaluate the integral $\oint_C \vec{F}\cdot d\vec{r}$, where C is the trace of the cylinder $x^2+y^2=1$ in the plane $x+y+z=1$.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -x^3 & z^3 \end{vmatrix} = 0\hat{i} - 0\hat{j} - 3(x^2+y^2)\hat{k}$$

Normal to the surface $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$$z = 1 - x - y$$

$$ds = \sqrt{1+1+1}\, dA = \sqrt{3}\, dA$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$$

$$= \iint_R \frac{-3(x^2+y^2)}{\sqrt{3}} \cdot \sqrt{3}\, dA$$

$$= -3\int_0^{2\pi}\int_0^1 r^2 \cdot r\, dr\, d\theta$$

$$= -3(2\pi) \cdot \frac{r^4}{4}\Big|_0^1$$

$$= -\frac{3\pi}{2}$$

Q:7 (12 points) Let $\vec{F}(x,y,z) = y^2z\mathbf{i} + x^3z^2\mathbf{j} + (z+2)^2\mathbf{k}$ and D is the region bounded by the cylinder $x^2+y^2=9$ and the planes $z=1,\ z=4$. Use divergence theorem to evaluate $\iint\limits_S \left(\vec{F}\cdot\hat{n}\right) ds$.

$$div\vec{F} = 0 + 0 + 2(z+2)$$

$$\iint\limits_S \vec{F}\cdot\hat{n}\, ds = \iiint\limits_D div\vec{F}\, dV$$

$$= \int_0^{2\pi}\int_0^3\int_1^4 (2z+4)\, r\, dz\, dr\, d\theta$$

$$= 2\pi \cdot \frac{r^2}{2}\Big|_0^3 \cdot (z^2+4z)\Big|_1^4$$

$$= 9\pi(32-5) = 9\pi(27) = 243\pi$$