Solution Midterm Exam 2 for Math 201 (073)

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Q.1: Identify the quadratic surface $4x^2+6y^2-3z^2-24x+12y+18z=17$. Find its intersections with the coordinate planes **(10 pts)**

Sol: $4x^2-24x+6y^2+12y-3z^2+18z=17$
$4\left(x^2-6x\right)+6\left(y^2+2y\right)-\left(3z^2-6z\right)=17$
$4\left(x^2-6x+9\right)+6\left(y^2+2y+1\right)-3\left(z^2-6z+9\right)=17+36+6-27$
$4\left(x-3\right)^2+6\left(y+1\right)^2-3\left(z-3\right)^2=32$
The surface is hyperboloid of one sheet.
Intersection with $xy-plane$: Put $z=0$, $4\left(x-3\right)^2+6\left(y+1\right)^2=27+32=59$, an ellipse.
Intersection with $xz-plane$: Put $y=0$, $4\left(x-3\right)^2-3\left(z-3\right)^2=32-6=26$, a hyperbola.
Intersection with $yz-plane$: Put $x=0$, $6\left(y+1\right)^2-3\left(z-3\right)^2=32-36=-4$, a hyperbola.

Q.2: Find the directional derivative of the function $f\left(x,y,z\right)=\dfrac{1}{x-2y+3z}$ at the point $(1,2,3)$ in the direction $v=\langle 1,2,3\rangle$. **(10 pts)**

Sol: $\nabla f\left(x,y,z\right)=<\dfrac{-1}{\left(x-2y+3z\right)^2},\dfrac{2}{\left(x-2y+3z\right)^2},\dfrac{-3}{\left(x-2y+3z\right)^2}>$

$\nabla f\left(1,2,3\right)=<\dfrac{-1}{36},\dfrac{2}{36},\dfrac{-3}{36}>=<\dfrac{-1}{36},\dfrac{1}{18},\dfrac{-1}{12}>$

Unite vector in the direction of v is $u=\dfrac{v}{|v|}=<\dfrac{1}{\sqrt{14}},\dfrac{2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}>$

$D_u f\left(1,2,3\right)=\nabla f\left(1,2,3\right)\cdot u=\dfrac{-1}{36\sqrt{14}}+\dfrac{2}{18\sqrt{14}}-\dfrac{3}{12\sqrt{14}}=-\dfrac{1}{84}\sqrt{14}.$

Q.3: Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the function $\tan\left(xyz\right)-2x^2y+y^2z=8$. **(10 pts)**

Sol: Let $F\left(x,y,z\right)=\tan\left(xyz\right)-2x^2y+y^2z-8=0$. Then

$\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{yz\sec^2\left(xyz\right)-4xy}{xy\sec^2\left(xyz\right)+y^2}$

$\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=\dfrac{xz\sec^2\left(xyz\right)-2x^2+2yz}{xy\sec^2\left(xyz\right)+y^2}.$

Q.4: (a) Write equation of the hyperboloid $x^2-y^2-z^2=1$ in spherical and cylindrical coordinates**(3+3 pts)**

Sol: For spherical coordinates, we shall use $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$.
$\left(\rho\sin\phi\cos\theta\right)^2-\left(\rho\sin\phi\sin\theta\right)^2-\left(\rho\cos\phi\right)=1$
$\rho^2\left(\sin^2\phi\cos^2\theta-\sin^2\phi\sin^2\theta-\cos^2\phi\right)=1$
$\rho^2\left(\sin^2\phi\cos 2\theta-\cos^2\phi\right)=1.$

For cylindrical coordinates, we shall use $x=r\cos\theta,\ y=r\sin\theta,\ z=z$.
$r^2\cos^2\theta-r^2\sin^2\theta-z^2=1$ OR $r^2\cos 2\theta-z^2=1$

(b) Find spherical and cylindrical coordinates of the point whose rectangular coordinates are $\left(0,2\sqrt{3},-2\right)$ **(2+2 pts)**

Spherical coordinates:

$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{0+12+4}=4,\ \theta=\tan^{-1}\left(\dfrac{2\sqrt{3}}{0}\right)=\dfrac{\pi}{2},\ \phi=\cos^{-1}\left(\dfrac{-2}{4}\right)=\dfrac{2}{3}\pi.$

Cylindrical coordinates

$r=\sqrt{x^2+y^2}=2\sqrt{3},\ \tan^{-1}\left(\dfrac{2\sqrt{3}}{0}\right)=\dfrac{\pi}{2},\ z=z.$

Q.5: Find the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{3x^2+2y^2}$ if exist or show that limit does not exist. **(10 pts)**

Sol: Let $x=0$ and $y\to 0$, then $\lim_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{3x^2+2y^2} = \lim_{y\to 0} \dfrac{\sin^2 y}{2y^2} = \dfrac{1}{2}$

Let $y=0$, and $x\to 0$, then $\lim_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{3x^2+2y^2} = \lim_{x\to 0} \dfrac{x^2}{3x^2} = \dfrac{1}{3}$.

Thus limit does not exist.

Q.6: Explain why $f(x,y) = \tan^{-1}(2x+3y)$ is differentiable at the point $P\left(0, \dfrac{1}{3}\right)$. Also find the linearization $L(x,y)$ of $f(x,y)$ at the point P. **(10 pts)**

Sol: $f\left(0,\dfrac{1}{3}\right) = \tan^{-1}(1) = \dfrac{1}{4}\pi$.

$\dfrac{\partial f}{\partial x} = \dfrac{2}{1+(2x+3y)^2}$ and $\dfrac{\partial f\left(0,\frac{1}{3}\right)}{\partial x} = \dfrac{2}{2} = 1$,

$\dfrac{\partial f}{\partial y} = \dfrac{3}{1+(2x+3y)^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{3}{2}$.

$L\left(0,\dfrac{1}{3}\right) = \dfrac{\pi}{4} + 1(x-0) + \dfrac{3}{2}\left(y-\dfrac{1}{3}\right) = \dfrac{\pi}{4} + x + \dfrac{3}{2}y - \dfrac{1}{2}$.

Q.7: The length l, width w and height h of a box are changing with time. If l is increasing at a rate of $3\ m/s$, w is decreasing at a rate of $2\ m/s$, and h is increasing at a rate of $1\ m/s$. Find the rate of change of the volume of the box when $l=10$, $w=8$, and $h=5$. **(10 pts)**

Sol: $l=10,\ \dfrac{dl}{dt}=3,\ w=8,\ \dfrac{dw}{dt}=-2,\ h=5,\ \dfrac{dh}{dt}=1.$

$V=lwh$, and $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial l}\dfrac{dl}{dt} + \dfrac{\partial V}{\partial w}\dfrac{dw}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = wh\dfrac{dl}{dt} + lh\dfrac{dw}{dt} + lw\dfrac{dh}{dt} = 40(3) + 50(-2) + 80(1) =$ 100.

Q.8: If $z=f(x,y)$, where $x=r\cos\theta$ and $y=r\sin\theta$. Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ and show that $\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$ **(10 pts)**

Sol: $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = -r\sin\theta\dfrac{\partial z}{\partial x} + r\cos\theta\dfrac{\partial z}{\partial y}$

$\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}\right)^2 + \left(-\sin\theta\dfrac{\partial z}{\partial x} + \cos\theta\dfrac{\partial z}{\partial y}\right) = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

Q.9: Find equation of tangent plane and normal line to the surface $x - z = 4\tan^{-1}(yz)$ at the point $\left(3+\pi, \frac{1}{3}, 3\right)$. **(10 pts)**

Sol: Let $F(x, y, z) = x - z - 4\tan^{-1}(yz) = 0$.

$\nabla F = < 1, \frac{-4z}{1+(yz)^2}, -1 - \frac{y}{1+(yz)^2} >$ and $\nabla F\left(3+\pi, \frac{1}{3}, 3\right) = \langle 1, -6, -\frac{7}{6} \rangle$.

Tangent plane $1(x - \pi - 3) - 6\left(y - \frac{1}{3}\right) - \frac{7}{6}(z - 3) = x - \pi - 6y - \frac{7}{6}z + \frac{5}{2} = 0$.

Normal line

$$\frac{x - \pi - 3}{1} = \frac{y - \frac{1}{3}}{-6} = \frac{z - 3}{\frac{-7}{6}}$$

Q.10: Find local minimum and local maximum values and saddle points of the function $f(x, y) = xy(1 - x - y)$. **(10 pts)**

Sol: $f(x, y) = xy - x^2y - xy^2$

$f_x = y - 2xy - y^2 = y(1 - 2x - y) = 0 \Rightarrow y = 0$ or $y = 1 - 2x$

$f_y = x - x^2 - 2xy = x(1 - 2y - x) = 0 \Rightarrow x = 0$ or $x = 1 - 2y$.

$y = 1 - 2x = 1 - 2(1 - 2y) = 1 - 2 + 4y$

$y = \frac{1}{3}$ and then $x = \frac{1}{3}$.

Thus the critical points are $(0,0)$, $(0,1)$, $(1,0)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 1 - 2x - 2y$,

$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$

$D(0,0) = 0 - 1 = -1 < 0$, so $(0,0)$ is a saddle point,

$D(1,0) = 0 - 1 = -1 < 0$, so $(1,0)$ is a saddle point,

$D(0,1) = 0 - 1 = -1 < 0$, so $(0,1)$ is a saddle point,

$D\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{4}{9} - \left(1 - \frac{4}{3}\right)^2 = \frac{1}{3} > 0$, and $f_{xx} = \frac{-2}{3} < 0$, so $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a local minimum.