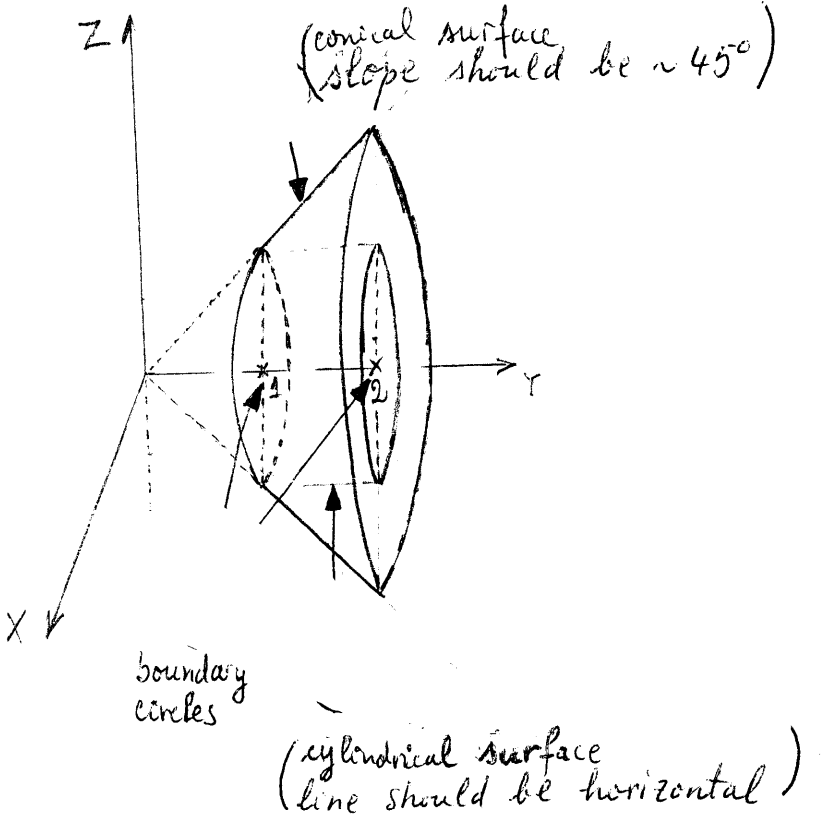
1. Sketch the region between the surface $y = \sqrt{x^2 + z^2}$ and $x^2 + z^2 = 1$ for $1 \leqslant y \leqslant 2$.



2. For the function $f(x, y) = \sin(x^2+y)$, compute the linear approximation of $f(1.02, -1.01)$.

$f(1,-1) = \sin(0) = 0$

$f_x(1,-1) = 2x\cos(x^2+y)\Big|_{\substack{x=1\\y=-1}} = 2$

$f_y(1,-1) = \cos(x^2+y)\Big|_{\substack{x=1\\y=-1}} = 1$

$dx = 1.02 - 1 = 0.02$ $\qquad$ $dy = -1.01 - (-1) = -0.01$

$L(1.02, -1.01) = 0 + 2(0.02) + 1 \cdot (-0.01) = 0.03$

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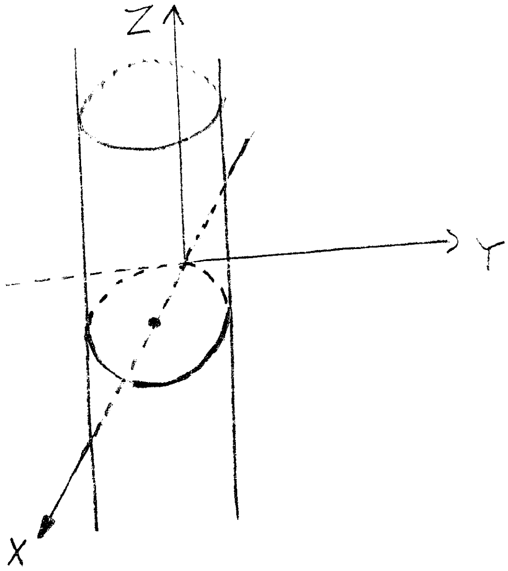
3. Transform the equation $\rho \sin\phi - \cos\theta = 0$ from spherical into cylindrical coordinates, and sketch the surface.

$$\left.\begin{array}{l} \textcircled{1}\ \rho\sin\phi = r \\ \textcircled{1}\ \cos\theta = \cos\theta \end{array}\right\} \Longrightarrow r = \cos\theta$$

It is a cylinder



$r^2 = r\cos\theta$

$x^2 + y^2 = x$

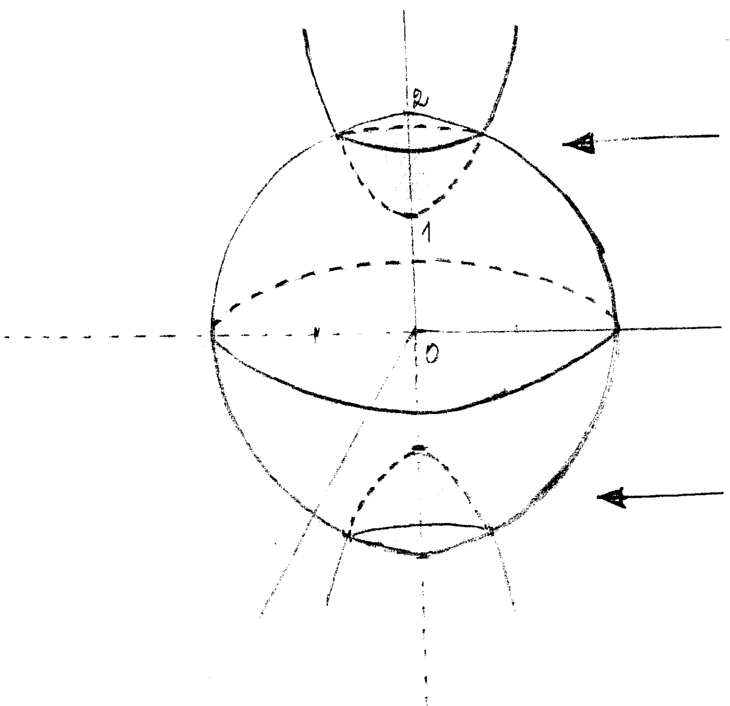
$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$

4. Name the boundary surfaces, and sketch the solid given in cylindrical coordinates by the inequalities $1 + r^2 \leqslant z^2 \leqslant 4 - r^2$.

$1 + r^2 = z^2 \longrightarrow x^2 + y^2 = z^2 - 1$ It is a 2-sheeted hyperboloid

$4 - r^2 = z^2 \longrightarrow x^2 + y^2 + z^2 = 4$ It is a sphere.

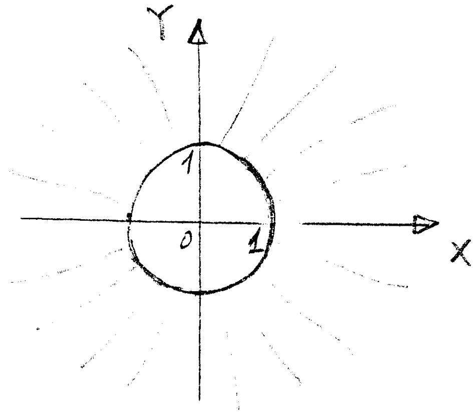


5. Consider the function $f(x, y) = \sqrt{x^2 + y^2 - 1}$:

(a) Describe in words and sketch the domain of f.

$\text{Domain}(f) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geqslant 1\}$

Are all the points outside and on the unit circle.



(b) Describe in words and sketch the graph of f.

$\text{Graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2 - 1}\}$

$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z \geqslant 0 \\ x^2 + y^2 = z^2 + 1 \end{array} \right\}$

It is the upper-half of a 1-sheeted hyperboloid.

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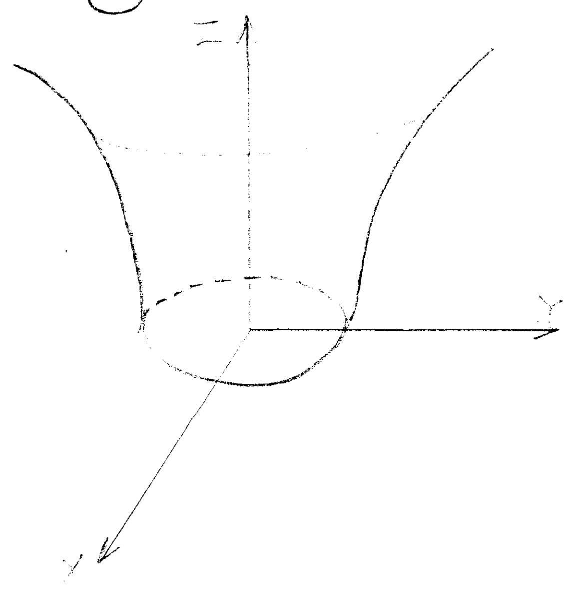

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6. For the function $f(x, y) = \tan(xy)$, compute the expression $f_y + f_{xy}$.

$$f_y(x,y) = \frac{x}{\cos^2(xy)}$$

$$f_{xy}(x,y) = \frac{1}{\cos^2(xy)} + x \cdot \frac{2\sin(xy)\cdot y}{\cos^3(xy)}$$

$$(\qquad\qquad)$$

$$f_y + f_{xy} = \frac{x}{\cos^2(xy)} + \frac{1}{\cos^2(xy)} + \frac{2xy \cdot \sin(xy)}{\cos^3(xy)}$$

7. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function z which is defined implicitly by the equation

$$x - z - \tan^{-1}(yz) = 0.$$

$$f(x,y,z) = x - z - \tan^{-1}(yz)$$

$$\frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)} = -\frac{1}{-1 - \frac{1}{1+y^2z^2}\cdot y} = \frac{1+y^2z^2}{1+y^2z^2+y}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)} = -\frac{-\frac{1}{1+y^2z^2}\cdot z}{-1 - \frac{1}{1+y^2z^2}\cdot y} = -\frac{z}{1+y^2z^2+y}$$

8. In the following problems: if the limit exists, compute it; if the limit does not exist, prove it.

(a) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{\sqrt{4+x^2+y^2+z^2}-2}$

$$\frac{x^2+y^2+z^2}{\sqrt{4+x^2+y^2+z^2}-2} = \sqrt{4+x^2+y^2+z^2}+2$$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \cdots = \lim_{(x,y,z)\to(0,0,0)} \left(\sqrt{4+x^2+y^2+z^2}+2\right) = 2+2=4.$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+8y^8}$.

Approach $(0,0)$ along $x=0$ (or $y=0$).

$$\lim_{(0,y)\to(0,0)} \frac{0\cdot y^4}{0+8y^8} = 0$$

Approach $(0,0)$ along $x=y^4$ (or $x=\text{const}\cdot y^4$).

$$\lim_{(y^4,y)\to(0,0)} \frac{y^4\cdot y^4}{y^8+8y^8} = \frac{1}{9}.$$

Then the limit does not exist.

9. Find the equation of the tangent plane and the normal line to the surface $z = e^{x-y^2}$ at the point $(1, -1, 1)$.

$$\nabla z = \left\langle e^{x-y^2} , -2y e^{x-y^2} \right\rangle$$

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$$\nabla z(1,-1) = \langle 1, 2 \rangle$$

The equation of the tangent plane is

$$z - 1 = (x-1) + 2(y+1) \Longrightarrow z = x + 2y + 2$$

The parametric equation of the normal line is:

$$\begin{cases} x = 1 + t \\ y = -1 + 2t \\ z = 1 - t \end{cases}$$

10. Given the surface defined implicitly by the equation $ye^{xz} + 2x + z = 3$, and the point $(1, 1, 0)$ on it, compute the directional derivative of z in the direction $\mathbf{u} = \langle 3, 4 \rangle$.

$$f(x,y,z) = ye^{xz} + 2x + z$$

$$\frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)} = -\frac{yze^{xz} + 2}{xye^{xz} + 1} \Longrightarrow \frac{\partial z}{\partial x}(1,1) = -\frac{2}{2} = -1$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)} = -\frac{e^{xz}}{xye^{xz} + 1} \Longrightarrow \frac{\partial z}{\partial y}(1,1) = -\frac{1}{2}$$

normalize u: $\longrightarrow$ $v = \frac{u}{|u|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

$$D_v z(1,1) = \frac{3}{5} \frac{\partial z}{\partial x}(1,1) + \frac{4}{5} \frac{\partial z}{\partial y}(1,1) = \frac{3}{5} \cdot (-1) + \frac{4}{5} \cdot \left(-\frac{1}{2}\right) = -1$$

11. Use the chain rule to find the partial derivative $\dfrac{\partial Y}{\partial t}$ for $Y = w \cdot \tan^{-1}(uv)$, where $u = r+s$, $v = s+t$, and $w = t+r$.

$$\frac{\partial Y}{\partial t}(s,t) = \frac{\partial Y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w} \cdot \frac{\partial w}{\partial t}$$

$$= \frac{wv}{1+u^2v^2} \cdot 0 + \frac{wu}{1+u^2v^2} \cdot 1 + \tan^{-1}(uv) \cdot 1$$

$u = r+s$
$v = s+t$
$w = t+r$

$$= \frac{(t+r)(r+s)}{1+(r+s)^2 \cdot (s+t)^2} + \tan^{-1}((r+s)(s+t))$$

12. Find the direction in which the function $f(x,y) = x^2 + xy + y^2$ increases the most rapidly at $(-1,1)$. Find the derivative of f in this direction.

$\nabla f(x,y) = \langle 2x+y,\ 2y+x \rangle$ 2

$\nabla f(-1,1) = \langle -1, 1 \rangle$ 1

normalize $\longrightarrow$ $u = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ 1

$D_u f(-1,1) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ 1