King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
MATH 201 – Calculus III
EXAM I
Semester I (092)

March 30, 2010 **Duration 120 minutes**

Student Name : ____________________

ID Number : ____________________

Section Number : ____________________

Instructions

- Write neatly and legibly. You may lose points for messy work.
- Show all your work.
- All types of calculators and mobiles are not allowed.

Question #	**Student Grade**	**Maximum Points**
Q.1		**16**
Q.2		**17**
Q.3		**16**
Q.4		**17**
Q.5		**17**
Q.6		**17**
TOTAL		**100**

1. (a) Find the length of the curve

$$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta);\quad 0 \le \theta \le \pi \quad (a > 0).$$

Sol.

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta)$$

So

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta$$
$$= a^2\theta^2$$

Thus the length of the curve is given by

$$\int_0^\pi \sqrt{a^2\theta^2}\, d\theta = \int_0^\pi a\theta\, d\theta = a^2\left[\frac{\theta^2}{2}\right]_0^\pi = \frac{a\pi^2}{2}$$

(b) A curve is defined by the parametric equations

$$x = t - e^t,\quad y = t + e^{-t}.$$

(i) Find $\dfrac{d^2y}{dx^2}$.

(ii) For which value of t is the curve concave upward?

Sol.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-e^{-t}}{1-e^t} = \frac{e^t - 1/e^t}{1-e^t}$$

$$= -e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{d}{dt}(e^{-t})}{\frac{dx}{dt}} = \frac{e^{-t}}{1-e^t}$$

The curve is concave upward

$$\frac{e^{-t}}{1-e^t} > 0 \Leftrightarrow 1-e^t > 0$$
$$\Leftrightarrow e^t < 1$$
$$\Leftrightarrow t \in (-\infty, 0)$$

or

$$t < 0$$

2. (a) Find equation of the tangent line of the curve with polar equation

$$r = 5 - 4\sin\theta \text{ at } \theta = \pi.$$

Sol. $x = r\cos\theta = (5 - 4\sin\theta)\cos\theta$

$y = r\sin\theta = (5 - 4\sin\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-4\cos\theta\sin\theta + 5\cos\theta - 4\sin\theta\cos\theta}{-4\cos^2\theta - 5\sin\theta + 4\sin^2\theta}$$

At $\theta = \pi$, slope $= \frac{5}{4}$ and $x = -5$, $y = 0$

Equation of tangent line is

$$y = \frac{5}{4}(x+5)$$

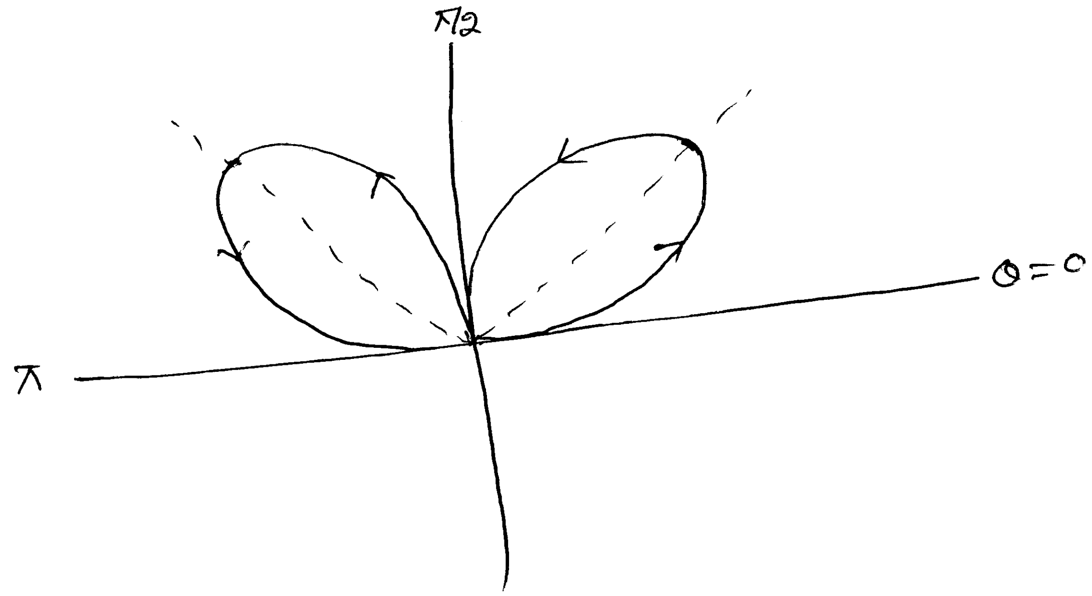
or $\boxed{4y - 5x = 5}$

(b) Sketch the polar curve $r = |\sin 2\theta|$, $0 \le \theta \le \pi$.

Sol.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	0	1	0	$\lvert -1 \rvert = 1$	0

(one each for each correct value)



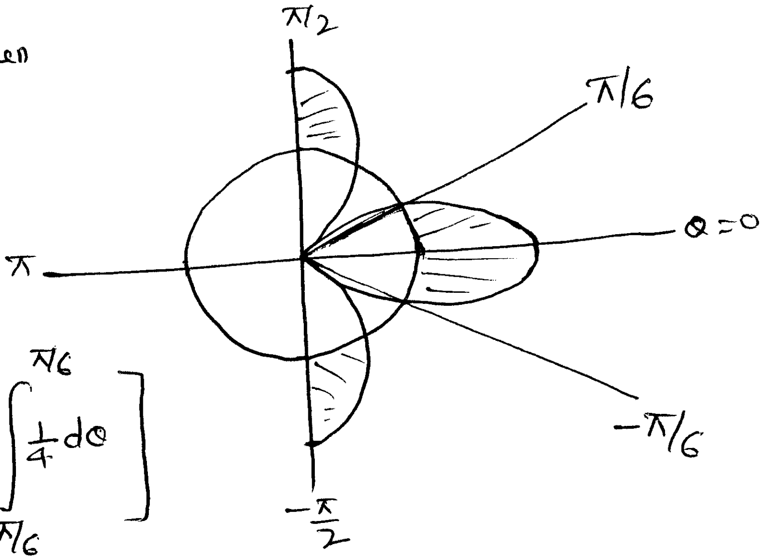
3. Find the area outside the curve $r = \frac{1}{2}$ and inside the curve $r = \cos 2\theta$ $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$.

<u>Sol</u>: Two curves intersect when

$$\frac{1}{2} = \cos 2\theta$$

$$\cos\frac{\pi}{3} = \cos 2\theta$$

$$\theta = \pm\frac{\pi}{6}$$



$$A = 2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 2\theta \, d\theta - \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1}{4} d\theta\right]$$

$$= 2\int_0^{\pi/6} \cos^2 2\theta \, d\theta - 2\int_0^{\pi/6} \frac{1}{4} d\theta$$

$$= 2\int_0^{\pi/6} \left(\frac{1+\cos 4\theta}{2}\right) - \frac{2\pi}{24}$$

$$= \left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/6} - \frac{\pi}{12}$$

$$= \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3} - \frac{\pi}{12}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

<u>Alt. Sol.</u>

$$A = \int_{-\pi/6}^{\pi/6} \cos^2 2\theta \, d\theta - \int_{-\pi/6}^{\pi/6} \frac{1}{4} d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1+\cos 4\theta}{2} d\theta - \left[\frac{1}{4}\theta\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{2}\frac{\pi}{3} - \frac{1}{4}\frac{\pi}{3} + \int_{-\pi/6}^{\pi/6} \frac{\cos 4\theta}{2}$$

$$= \frac{\pi}{12} + \left(\frac{\sin 4\theta}{8}\right)_{-\pi/6}^{\pi/6} = \frac{\pi}{12} + \frac{2}{8}\sin\frac{4\pi}{6}$$

$$= \frac{\pi}{12} + \frac{1}{4}\sin\frac{2\pi}{3} = \frac{\pi}{14} + \frac{\sqrt{3}}{8}$$

4. (a) Prove that the mid-point of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

Sol. Let $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ be a point on the line joining P_1 and P_2 for $t = \frac{1}{2}$

$$d(M, P_1) = \left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 + \left(\frac{z_1-z_2}{2}\right)^2$$

$$d(M, P_2) = \left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2 + \left(\frac{z_2-z_1}{2}\right)^2$$

$$d(M, P_1) = d(M, P_2)$$

So M is a mid point of P_1, P_2

(b) Find the equation of the sphere if one of its diameters has end-points $(2, 1, 4)$ and $(4, 3, 10)$.

Sol. centre is $\left(\frac{4+2}{2}, \frac{3+1}{2}, \frac{10+4}{2}\right) = (3, 2, 7)$

Radious $= \frac{1}{2}\sqrt{(4-2)^2 + (3-1)^2 + (10-4)^2} = \frac{\sqrt{4+4+36}}{2}$

$= \frac{\sqrt{44}}{2} = \sqrt{11}$

The equation of sphere is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$$

(c) If the angle between two unit vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$, then find the value of $|2\vec{a} - 3\vec{b}|$.

Sol. $|2\vec{a} - 3\vec{b}|^2 = (2\vec{a} - 3\vec{b}) \cdot (2\vec{a} - 3\vec{b})$

$= 4|\vec{a}|^2 - 12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2$

$= 4 - 12|\vec{a}||\vec{b}|\cos\frac{\pi}{3} + 9$

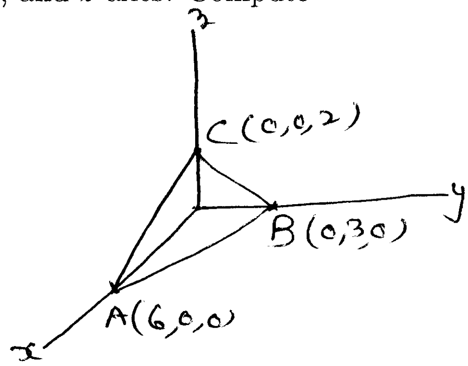
$= 13 - 12 \cdot 1 \cdot 1 \cdot \frac{1}{2}$

$= 13 - 6$

$= 7$

$$\boxed{|2\vec{a} - 3\vec{b}| = \sqrt{7}}$$

5. (a) Given the plane P defined by equation $x + 2y + 3z = 6$, we denote by A, B and C the intersection points of the plane P with the x-, y-, and z-axes. Compute the area of triangle ABC.



Sol. The point A has coordinate (6,0,0)
" " B " " (0,3,0)
" " C " " (0,0,2)

$$\vec{AB} = \langle -6, 3, 0 \rangle \; ; \; \vec{AC} = \langle -6, 0, 2 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -6 & 3 & 0 \\ -6 & 0 & 2 \end{vmatrix} = \langle 6, 12, 18 \rangle$$

$$\text{Area of triangle } ABC = \frac{|\vec{AB} \times \vec{AC}|}{2} = \frac{1}{2}\sqrt{6^2+12^2+18^2}$$

$$= \frac{6}{2}\sqrt{1+4+9} = 3\sqrt{14}$$

(b) Determine whether the points $P(1, 0, 1)$, $Q(2, 4, 6)$, $R(3, -1, 2)$ and $S(6, 2, 8)$ lie on the same plane.

Sol. $\vec{PQ} = \langle 1, 4, 5 \rangle$, $\vec{PR} = \langle 2, -1, 1 \rangle$ and $\vec{PS} = \langle 5, 2, 7 \rangle$

$$\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = \begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix}$$

$$= 1(-7-2) - 4(14-5) + 5(4+5)$$

$$= -9 - 36 + 45$$

$$= 0$$

Hence these vectors lie on the same plane.

6. (a) Find the equation of the plane passing through the origin O, which is parallel to the z-axis and is perpendicular to the plane $3x - 2y + z = 4$.

Sol. Let $ax + by + cz = 0$ be the equation of the plane. Then its normal vector is $\vec{n} = \langle a, b, c \rangle$

$\vec{n}$ is perpendicular to $\langle 0, 0, 1 \rangle$ and to $\langle 3, -2, 1 \rangle$

$\Rightarrow \vec{n}$ is parallel to $\langle 0, 0, 1 \rangle \times \langle 3, -2, 1 \rangle$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \langle 2, 3, 0 \rangle$$

The equation of plane is

$$\boxed{2x + 3y = 0}$$

(b) Find the point of the intersection (if any) of

$$L_1 : x = -6t, y = 1 + 9t, z = -3t$$
$$L_2 : x = 1 + 2s, y = 4 - 3s, z = s.$$

Sol. Direction vectors ~~and~~ of the lines are $\vec{a} = \langle -6, 9, -3 \rangle$ and $\vec{b} = \langle 2, -3, 1 \rangle$.

We have $\vec{a} = -3\vec{b}$

Hence the lines are parallel, so if they intersect then L_1 must be same as L_2

The point $(0, 1, 0)$ is a point on L_1. If it is a point on L_2 then $1 + 2s = 0$, $4 - 3s = 0$, $s = 0$ for some s. This is not possible. Hence lines have no intersection.

OR At a pt of intersection

$-6t = 1 + 2s$, $1 + 9t = 4 - 3s$, $-3t = s$

From Ist and 3rd eqn,

$2s = 1 + 2s$ ① $\Rightarrow 1 = 0$ which is impossible

So lines have no point of intersection.