King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Solution Final Exam for Math 131 (073)

Date: August 26, 2008 Instructor: Dr. Muhammad Yousuf

Q.1: A company makes car stereos. The manufacturing cost for each stereo is \$45. The company has fixed costs of \$4150 per month. Find the linear cost function and determine the number of stereos company can make for a total cost of \$10,000. **(10 pts)**

Sol: The cost function is $C(x) = 45x + 4150$.

When $C = 10,000$, then $10,000 = 45x + 4150$ gives $x = \dfrac{10000 - 4150}{45} = 130$.

Q.2: For the parabola $y = f(x) = -x^2 + 7x - 6$, find: (a) the vertex, (b) the y-intercept, and (c) the x-intercepts. **(10 pts)**

Sol: $y = -\left(x^2 - 7x + \dfrac{49}{4}\right) + \dfrac{49}{4} - 6 = -\left(x - \dfrac{7}{2}\right)^2 + \dfrac{25}{4}$

(a) The vertex is $\left(\dfrac{7}{2}, \dfrac{25}{4}\right)$

(b) $y - i$intercept is $x = 0$ and $y = -6$.

(c) -$i$ntercepts are, $y = 0$ and $x^2 - 7x + 6 = 0$, gives $x = 1$, and $x = 6$.

Q.3: Solve the following system of equations algebraically. (If the system does not have a solution, then say so or if it has more than one unique solution, then describe the solutions.) **(10 pts)**

$$\begin{aligned} x + y + z &= 2 \qquad I \\ x + 2y + 3z &= 4 \qquad II \\ x + 3y + 5z &= 6 \qquad III \end{aligned}$$

Sol: Subtract I from II, $y + 2z = 2$ and subtract II from III, $y + 2z = 2$.
$y = 2 - 2z$ and from I, $x = 2 - y - z = 2 - 2 + 2z - z = z$.
The solution is $x = z,\ y = 2 - 2z$, and z is arbitrary.

Q.4: Solve the nonlinear system of equations $\begin{cases} x - y - 1 = 0 & I \\ y = \sqrt{x + 5} & II \end{cases}$. Check which solution is a correct solution.

(10 pts)

Sol: $x - 1 = y = \sqrt{x + 5}$
$(x - 1)^2 = x + 5$
$x^2 - 2x + 1 - x - 5 = 0$
$x^2 - 3x - 4 = 0$, Solution is: $x = -1,\ x = 4$
For $x = -1,\ I$ gives $y = -2$ and II gives $y = 2$. So $x = -1$ is not a solution.
For $x = 4,\ I$ gives $y = 3$ and II gives $y = 3$. So $x = 4$ is a solution.

Q.5: Determine the effective rate equivalent to an annual rate of $7\frac{3}{4}\%$ compounded continuously. **(5 pts)**

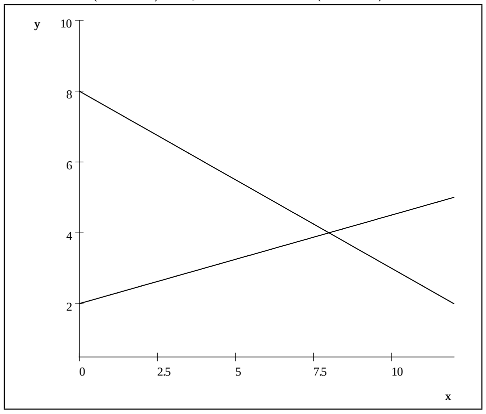
Sol: Effective rate is $r_e = e^r - 1 = e^{0.0775} - 1 = 0.080582 = 8.0582\%$.

Q.6: Use linear programming method to maximize the objective function $Z = 2x + 4y$, subject to the constraints $\begin{matrix} x - 4y \le -8 \\ x + 2y \le 16 \\ x, y \ge 0 \end{matrix}$. Write formulas for all values of x and y if Z is maximum at two corner points.

(12 pts)

Sol: $x - 4y \le -8$ gives $-x + 4y \ge 8$. The points of intersection of two lines is $(8, 4)$. Thus the corner points are $A(0, 8)$, $B(8, 4)$, and $C(0, 2)$.
$Z(A) = 32,\ Z(B) = 32,\ Z(C) = 8$.

The objective function is maximum at two points $A(0,8)$ and $B(8,4)$. Thus it is maximum at $x = 0t + 8(1-t)$, $y = 8t + 4(1-t)$, $0 \le t \le 1$.



Q.7: Suppose a person deposits \$1000 in a savings account in the begining of every six months. What is the value of the account at the end of five years if interest is at a rate of 10% compounded semiannually? **(10 pts)**

Sol: $$S = R(1+r)\left(\frac{(1+r)^n - 1}{r}\right) = 1000(1+0.05)\left(\frac{(1+0.05)^{10} - 1}{0.05}\right) = 13207.$$

Q.8: Find the sum of the geometric series: $1 + 2 + 4 + 8 + 16 \cdots + 2^{15}$. **(10 pts)**

Sol: $a = 1,\ r = 2,\ n = 16,$

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{2^{16} - 1}{2 - 1} = 65535.$$

Q.9: Write the system into matrix form and solve by the method of reduction **(13 pts)**

$$\begin{aligned} x - y - 3z &= -5 \\ 2x - y - 4z &= -8 \\ x + y - z &= -1 \end{aligned}$$

Sol: $$\begin{bmatrix} 1 & -1 & -3 & -5 \\ 2 & -1 & -4 & -8 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \\ -R_1 + R_3 \end{matrix} \begin{bmatrix} 1 & -1 & -3 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 4 \end{bmatrix} -2R_2 + R_3 \begin{bmatrix} 1 & -1 & -3 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\tfrac{-1}{2}R_3 \begin{bmatrix} 1 & -1 & -3 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$z = 0,\ y = 2,$ and $x = -5 + y + 3z = -5 + 2 + 0 = -3.$

The solution is $(-3, 2, 0)$.

Q.10: Find the number of distinguishable horizontal arrangements of all the letters in MATHEMATICAL. **(10 pts)**

Sol: $A's = 3,\ M's = 2,\ T's = 2,\ H's = 1,\ E's = 1,\ I's = 1,\ C's = 1,\ L's. = 1$

The number of distinguishable arrangements is $\dfrac{12!}{3!2!2!1!1!1!1!1!} = 19958400.$

Q.11: An urn contains ten marbles numbered 1 through 10. If two marbles are randomly drawn in succession without replacement, determine the probability that: **(10 pts)**

(a) the first marble drawn shows 1 and the second shows 2;

(b) both show an odd number.

Sol: $P = \dfrac{1}{10}\dfrac{1}{9} = \dfrac{1}{90}.$

$P = \dfrac{5}{10}\dfrac{4}{9} = \dfrac{20}{90} = \dfrac{2}{9}.$

Q.12: If $P(E)=\frac{1}{3}, P(F')=\frac{2}{5}$, and $P(E\cap F)=\frac{1}{5}$, find $P(E\mid F)$. and $P(E\cup F)$. **(10 pts)**

Sol: $P(F)=1-P(F')=1-\frac{2}{5}=\frac{3}{5}$,

$$P(E\mid F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{5}}{\frac{3}{5}}=\frac{1}{3}.$$

$$P(E\cup F)=P(E)+P(F)-P(E\cap F)=\frac{1}{3}+\frac{3}{5}-\frac{1}{5}=\frac{1}{3}+\frac{2}{5}=\frac{11}{15}.$$

Q.13: If $P(E\mid F)=\frac{1}{2}$, $P(E\cup F)=\frac{9}{10}$, and $P(E\cap F)=\frac{2}{5}$, determine if E and F are independent or dependent. **(10 pts)**

Sol: $P(E\mid F)=\frac{P(E\cap F)}{P(F)}\Rightarrow P(F)=\frac{P(E\cap F)}{P(E\mid F)}=\frac{\frac{2}{5}}{\frac{1}{2}}=\frac{4}{5}$,

$$P(E\cup F)=P(E)+P(F)-P(E\cap F)\Rightarrow P(E)=P(E\cup F)-P(F)+P(E\cap F)=\frac{9}{10}-\frac{4}{5}+\frac{2}{5}=\frac{1}{2}.$$

$P(E)P(F)=\frac{4}{5}\frac{1}{2}=\frac{2}{5}=P(E\cap F)$. The events are independent.

Q.14: An urn contains 4 red and 6 green marbles. Three marbles are successively drawn at random without replacement and the number of red marbles, X, is observed. Construct the probability histogram for X. **(15 pts)**

Sol: The random variable X has values $X=0,1,2,3$

$P(X=0)=\frac{6}{10}\frac{5}{9}\frac{4}{8}=\frac{120}{720}=0.1667$ (GGG)

$P(X=1)=\frac{4}{10}\frac{6}{9}\frac{5}{8}+\frac{6}{10}\frac{4}{9}\frac{5}{8}+\frac{6}{10}\frac{5}{9}\frac{4}{8}=\frac{360}{720}=0.5000$ $(RGG$ or GRG or $GGR)$

$P(X=2)=\frac{4}{10}\frac{3}{9}\frac{6}{8}+\frac{4}{10}\frac{6}{9}\frac{3}{8}+\frac{6}{10}\frac{4}{9}\frac{3}{8}=\frac{216}{720}=0.3000$ $(RRG$ or RGR or $GRR)$

$P(X=3)=\frac{4}{10}\frac{3}{9}\frac{2}{8}=\frac{24}{720}=0.0333$ (RRR)

OR we can also solve this problem as:

${}_{10}C_3=120$

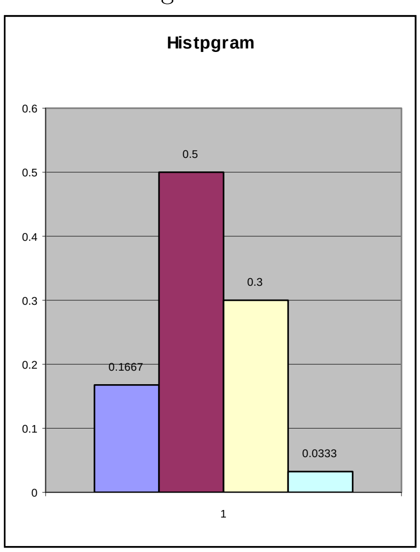
$P(X=0)=\frac{{}_4C_0\cdot{}_6C_3}{{}_{10}C_3}=\frac{(1)\,(20)}{120}=0.1667$

$P(X=1)=\frac{{}_4C_1\cdot{}_6C_2}{{}_{10}C_3}=\frac{(4)\,(15)}{120}=0.5000$

$P(X=2)=\frac{{}_4C_2\cdot{}_6C_1}{{}_{10}C_3}=\frac{(6)\,(6)}{120}=0.3000$

$P(X=3)=\frac{{}_4C_3\cdot{}_6C_0}{{}_{10}C_3}=\frac{(4)\,(1)}{120}=\frac{4}{120}=0.0333$

The histogram is



Q.15: Suppose X is a binomially distributed random variable such that $\mu = 2$ and $\sigma^2 = \frac{3}{2}$. Find:

(a) $P(X = 2)$
(b) $P(X \geq 3)$
(c) $P(X < 7)$ **(2+4+4 pts)**

Sol: $\mu = np = 2$, and $\sigma^2 = npq = \frac{3}{2} \Rightarrow q = \frac{3}{4}$ and $p = \frac{1}{4}$, $n = 8$.

(a) $P(X = 2) = \binom{8}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 = 0.31146.$

(b) $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$$= 1 - \binom{8}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^8 - \binom{8}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^7 - \binom{8}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 = 0.321\,46$$

(c) $P(X < 7) = 1 - P(X = 8) = 1 - \binom{8}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^0 = 0.99998.$

Q.16: The scores on an examination are normally distributed with $\mu = 75$ and $\sigma^2 = 10$. Find percentage of the scores that lies between 70 and 90. **(10 pts)**

Sol: $z_1 = \frac{70 - 75}{10} = -0.5$ and $z_1 = \frac{90 - 75}{10} = 1.5$

$P(-0.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -0.5) = 0.9332 - 0.3085 = 0.6247.$

Q.17: If Z has a standard normal distribution, find $P(-3 \leq Z \leq -0.5)$. **(10 pts)**

Sol: $P(-3 \leq Z \leq -0.5) = P(Z < -0.5) - P(Z < -3) = 0.3085 - 0.0013 = 0.3072.$

Q.18: If The scores of the top ten finishers in a recent LPGA Valley of the Stars Tournament are listed below. (Source: Los Angeles Times) Find the mean score and sample standard deviation. **(10 pts)**

71 67 67 72 76 73 68

Sol: $x = 71 + 67 + 67 + 72 + 76 + 73 + 68 = 494$

$\bar{x} = \frac{494}{7} = 70.571$

$x^2 = (71)^2 + (67)^2 + (67)^2 + (72)^2 + (76)^2 + (73)^2 + (68)^2 = 34932$

$s^2 = \frac{34932 - 7\,(70.571)^2}{6} = 11.690$

$s = \sqrt{11.690} = 3.4191.$

Q.19: For the following data set, approximate the sample standard deviation of phone calls per day, **(15 pts)**

Phone calls (per day)	Frequency	x	fx	x^2	fx^2
8 - 11	18	9.5	171.0	90.25	1624.50
12 - 15	23	13.5	310.5	182.25	4191.75
16 - 19	38	17.5	665.0	306.25	11637.50
20 - 23	47	21.5	1010.5	462.25	21725.75
24 - 27	32	25.5	816.0	650.25	20808.00

$n = \sum f = 158$, $\sum fx = 2973$, $\sum fx^2 = 59987.5$

$\bar{x} = \frac{\sum fx}{n} = \frac{2973}{158} = 18.816$, $s^2 = \frac{\sum fx^2 - n\,(\bar{x})^2}{n-1} = \frac{59987.5 - 158\,(18.816)^2}{157} = 25.789.$

$s = \sqrt{25.789} = 5.0783.$