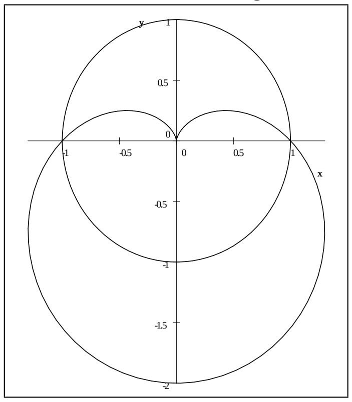
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Solution Final Exam for Math 201 (073)

Date: August 26, 2008 Instructor: Dr. Muhammad Yousuf

Q.1: Find area of the region that lies inside $r = 1 - \sin\theta$ and outside $r = 1$. **(10 pts)**



$A = \int_{\pi}^{2\pi} \frac{1}{2}\left((1-\sin\theta)^2 - 1\right) d\theta = \frac{1}{4}\pi + 2$

Q.2: Find equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$. **(10 pts)**

$n_1 = \langle 1, 0, -1\rangle,\ n_2 = \langle 0, 1, 2\rangle$ and $n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = i - 2j + k.$

$n_3 = \langle 1, 1, -2\rangle$ and $(n_1 \times n_2) \times n_3 = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3i + 3j + 3k.$

A points on the line of intersection is $(1, 3, 0)$ and equation of the required plane is $3(x-1) + 3(y-3) + 3(z-0) = 3x + 3y + 3z - 12 = 0$ or $x + y + z = 4$.

Q.3: Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u, and v are differentiable, $u(1,0) = 2$, $u_s(1,0) = -1$, $u_t(1,0) = 6$, $v(1,0) = 3$, $v_s(1,0) = 5$, $v_t(1,0) = 4$, $F_u(2,3) = -1$,and $F_v(2,3) = 10$. Find $W_s(1,0)$ and $W_t(1,0)$. **(10 pts)**

$W_s(s,t) = F_u(u(s,t), v(s,t)).u_s(s,t) + F_v(u(s,t), v(s,t)).v_s(s,t)$

and $W_s(1,0) = F_u(u(1,0), v(1,0))\, u_s(1,0) + F_v(u(1,0), v(1,0))\, v_s(1,0) = F_u(2,3).(-1) + F_v(2,3).5 = (-1)(-1) + 10(5) = 51.$

$W_t(s,t) = F_u(u(s,t), v(s,t)).u_t(s,t) + F_v(u(s,t), v(s,t)).v_t(s,t)$

and $W_t(1,0) = F_u(u(1,0), v(1,0))\, u_t(1,0) + F_v(u(1,0), v(1,0))\, v_t(1,0) = F_u(2,3).(6) + F_v(2,3).4 = (-1)(6) + 10(4) = 34.$

Q.4: Show that the sum of $x-$, $y-$, and $z-$ intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant. **(10 pts)**

Let (x_o, y_o, z_o) be any point on tyhe surface. Then $\sqrt{x_o} + \sqrt{y_o} + \sqrt{z_o} = \sqrt{c}$.

$\nabla f(x,y,z) = \langle \frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}} \rangle$ and $n = \nabla f(x_o, y_o, z_o) = \langle \frac{1}{2\sqrt{x_o}}, \frac{1}{2\sqrt{y_o}}, \frac{1}{2\sqrt{z_o}} \rangle$

Equation of the tangent plane is $\frac{1}{2\sqrt{x_o}}(x - x_o) + \frac{1}{2\sqrt{x_o}}(y - y_o) + \frac{1}{2\sqrt{x_o}}(z - z_o) = 0$

$\frac{x}{\sqrt{x_o}} + \frac{y}{\sqrt{y_o}} + \frac{z}{\sqrt{z_o}} = \sqrt{x_o} + \sqrt{y_o} + \sqrt{z_o} = \sqrt{c}.$

$x - intercept:\ x = \sqrt{x_o c},\ y - intercept:\ y = \sqrt{y_o c},\ z - intercept:\ z = \sqrt{z_o c}$

Sum is $\sqrt{x_o c} + \sqrt{x_o c} + \sqrt{x_o c} = \sqrt{c}\left(\sqrt{x_o} + \sqrt{y_o} + \sqrt{z_o}\right) = \sqrt{c}\sqrt{c} = c.$

Q.5: Find volume of the parallelepiped determined by the vectors $\mathbf{a} = \langle 6, 3, -1\rangle$, $\mathbf{b} = \langle 0, 1, 2\rangle$ and $\mathbf{c} = \langle 4, -2, 5\rangle$. **(10 pts)**

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 82.$$

Q.6: Find linear approximation of the function $f(x, y) = \sqrt{20 - 7x^2 - y^2}$ at the point $(1, 2)$ **(10 pts)**

$f_x(x, y) = \frac{-14x}{2\sqrt{20-7x^2-y^2}}$ and $f_y(x, y) = \frac{-2y}{2\sqrt{20-7x^2-y^2}}$
$f_x(1, 2) = \frac{-14}{6} = \frac{-7}{3}$, $f_y(1, 2) = \frac{-4}{6} = \frac{-2}{3}$.
$L(x_o, y_o) = f(x_o, y_o) + f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$
$L(1, 2) = 3 - \frac{7}{3}(x-1) - \frac{2}{3}(y-2) = \frac{20}{3} - \frac{2}{3}y - \frac{7}{3}x.$

Q.7: Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$. **(10 pts)**

Minimize $d = (x-1)^2 + (y-2)^2 + (z-3)^2 = (x-1)^2 + (y-2)^2 + (4-x+y-3)^2 = 2x^2 - 2y - 2xy - 4x + 2y^2 + 6$
Let $f(x, y) = 2x^2 - 2y - 2xy - 4x + 2y^2 + 6$
$f_x = 4x - 2y - 4$, and $f_y = 4y - 2x - 2$.
$f_x = 0$ and $f_y = 0$ gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$. Thus the only critical point is $\left(\frac{5}{3}, \frac{4}{3}\right)$.
For $x = \frac{5}{3}$ and $y = \frac{4}{3}$, $z = 4 - \frac{5}{3} + \frac{4}{3} = \frac{11}{3}$
So the point on the plane that is closest to $(1, 2, 3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

Q.8: Use Lagrange Multipliers to find the points on the sphere $x^2 + y^2 + z^2 = 22$ that are closest to and farthest from the point $(3, 1, -1)$. **(10 pts)**

Minimize $d = f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 22$.
$\nabla f = \lambda\nabla g \Rightarrow 2(x-3) = 2x\lambda,\ 2(y-1) = 2y\lambda,\ 2(z+1) = 2z\lambda.$
$2x(1-\lambda) = 6$ or $x = \frac{3}{1-\lambda}$, similarly $y = \frac{1}{1-\lambda}$, $z = \frac{-1}{1-\lambda}$.
$\frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 22 \Rightarrow (1-\lambda)^2 = \frac{11}{22} \Rightarrow \lambda = 1 \pm \sqrt{\frac{1}{2}} = 1 \pm \frac{1}{\sqrt{2}}.$
The corresponding points are $\left(\pm 3\sqrt{2}, \pm\sqrt{2}, \mp\sqrt{2}\right)$. The points that is farthest from $(3, 1, -1)$ is $\left(-3\sqrt{2}, -\sqrt{2}, \sqrt{2}\right)$ and the points that is closest to the point $(3, 1, -1)$ is $\left(3\sqrt{2}, \sqrt{2}, -\sqrt{2}\right)$.

Q.9: Evaluate the integral $\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x\sqrt{1 + \cos^2 x}\, dx dy$ (Hint: Change order of integration). **(10 pts)**

$0 \le y \le 1,\ \sin^{-1} y \le x \le \frac{\pi}{2}$.
$\int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x\sqrt{1 + \cos^2 x}\, dy dx = \int_0^{\frac{\pi}{2}} \sin x \cos x\sqrt{1 + \cos^2 x} dx = \frac{1}{2}\int_1^2 u^{\frac{1}{2}} du = \frac{2}{3}\sqrt{2} - \frac{1}{3}$

Q:10: Sketch the region and use polar coordinates to combine the integrals into one integral and then evaluate the integral $\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy dy dx + \int_1^{\sqrt{2}} \int_0^x xy dy dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy dy dx.$ **(12 pts)**

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy dy dx + \int_1^{\sqrt{2}} \int_0^x xy dy dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy dy dx = \int_0^{\frac{\pi}{4}} \int_1^2 r\cos\theta r \sin\theta r dr d\theta = \int_1^2 r^3 dr \int_0^{\frac{\pi}{4}} \cos\theta \sin\theta d\theta =$$
$\left(4 - \frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{15}{16}$

Q.11: Find volume of a tetrahedron enclosed by the planes $x = 0,\ y = 0,\ z = 0,$ and $2x + 2y + z = 4$. **(10 pts)**

$$V = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dzdydx = \tfrac{8}{3}.$$

Q.12: Use spherical coordinates to evaluate $\iiint\limits_E x^2 dV$, where E is the solid region bounded by the hemispheres $x = \sqrt{9 - y^2 - z^2}$ and $x = \sqrt{36 - y^2 - z^2}$and the $yz - plane$. **(13 pts)**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{x} \int_3^{6} \rho^2 \sin^2(\phi) \cos^2(\theta)\, \rho^2 \sin(\phi)\, d\rho d\phi d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta \int_0^{x} \left(1 - \cos^2(\phi)\right) \sin(\phi)\, d\phi \int_3^{6} \rho^4 d\rho = \tfrac{5022\pi}{5}.$$