3

General Case:

In general,

If $\lambda = \alpha + i\beta$ is a complex eigen value, and K is an eigen vector corresponding to λ, then two solutions are $Ke^{\lambda t}$ and $\bar{K}e^{\bar{\lambda} t}$.

By Super-position principle, we have solutions:

$$X_1 = \tfrac{1}{2}\left(Ke^{\lambda t} + \bar{K}e^{\bar{\lambda} t}\right), \quad X_2 = \tfrac{i}{2}\left(-Ke^{\lambda t} + \bar{K}e^{\bar{\lambda} t}\right)$$

So

$$X_1 = \tfrac{1}{2}\left(Ke^{\lambda t} + \bar{K}e^{\bar{\lambda} t}\right) = \tfrac{1}{2}\left(Ke^{(\alpha+i\beta)t} + \bar{K}e^{(\alpha-i\beta)t}\right)$$

$$= \tfrac{1}{2}\left[Ke^{\alpha t}e^{i\beta t} + \bar{K}e^{\alpha t}e^{-i\beta t}\right]$$

$$= \tfrac{1}{2}e^{\alpha t}\left[Ke^{i\beta t} + \bar{K}e^{-i\beta t}\right]$$

$$= \tfrac{1}{2}e^{\alpha t}\left[K(\cos\beta t + i\sin\beta t) + \bar{K}(\cos\beta t - i\sin\beta t)\right]$$

$$= \tfrac{1}{2}e^{\alpha t}\left[(K+\bar{K})\cos\beta t + (K-\bar{K})i(\sin\beta t)\right]$$

$$X_2 = \tfrac{i}{2}\left[-Ke^{\lambda t} + \bar{K}e^{\bar{\lambda} t}\right] = \tfrac{i}{2}\left[-Ke^{(\alpha+i\beta)t} + \bar{K}e^{(\alpha-i\beta)t}\right]$$

$$= \tfrac{i}{2}\left[-Ke^{\alpha t}e^{i\beta t} + \bar{K}e^{\alpha t}e^{-i\beta t}\right]$$

$$= \tfrac{i}{2}e^{\alpha t}\left[-Ke^{i\beta t} + \bar{K}e^{-i\beta t}\right]$$

$$= \tfrac{i}{2}e^{\alpha t}\left[-K(\cos\beta t + i\sin\beta t) + \bar{K}(\cos\beta t - i\sin\beta t)\right]$$

$$\therefore X_1 = \frac{e^{\alpha t}}{2}\left[(K+\bar{K})\cos\beta t + (K-\bar{K})i\sin\beta t\right]$$

$$X_2 = \frac{ie^{\alpha t}}{2}\left[(\bar{K}-K)\cos\beta t - i(K+\bar{K})\sin\beta t\right]$$

Now, $K+\bar{K}$ and $i(\bar{K}-K)$ are real. ← [ask the students to obtain that $i(\bar{K}-K)$ is real, $K+\bar{K}$ ---]

Define $B_1 = \tfrac{1}{2}(K+\bar{K})$, $B_2 = \tfrac{i}{2}(-K+\bar{K})$. Then

$$\left.\begin{aligned} X_1 &= (B_1\cos\beta t - B_2\sin\beta t)e^{\alpha t} \\ X_2 &= (B_2\cos\beta t + B_1\sin\beta t)e^{\alpha t} \end{aligned}\right\} \text{ and } X = c_1X_1 + c_2X_2.$$

Note that $B_1 = \frac{1}{2}(K + \bar{K}) = Re(K)$

$B_2 = \frac{i}{2}(-K + \bar{K}) = Im(K)$.

So,
$$X_1 = \left(Re(K) \cos \beta t - Im(K) \sin \beta t\right) e^{\alpha t}$$

$$X_2 = \left(Im(K) \cos \beta t + Re(K) \sin \beta t\right) e^{\alpha t}$$

Now Go to the example in your note and apply the above:

Recall, $\lambda = \underset{\alpha}{5} + \underset{\beta}{2}i$, $K = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$

$Re(K) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Im(K) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$$\therefore \quad X_1 = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} -2 \\ 0 \end{bmatrix} \sin 2t \right) e^{5t}$$

$$X_2 = \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t \right) e^{5t}$$

and the general Solution is $X = c_1 X_1 + c_2 X_2$