Section 7.3 *The Eigenvalue method for Linear Systems*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. see the relation between eigenvalues (& eigenvectors) of the matrix A with solutions of homogeneous linear system $X' = AX$
2. solve homogeneous systems $X' = AX$ if
   - A has distinct real eigenvalues
   - A has distinct complex eigenvalues
   - A has both distinct real and complex eigenvalues

Observation: Relation of eigenvalues (& eigenvectors) with solutions of homogeneous systems

- Consider a homogeneous linear system

$$X' = AX \qquad (*)$$

- Question: When a vector of the form $X = \mathbf{v}e^{\lambda t}$ will be a solution of (*)?
- Answer: $X = \mathbf{v}e^{\lambda t} \Rightarrow X' = \mathbf{v}\lambda e^{\lambda t}$

  Using X, X' in (*) we see that

  $\mathbf{v}\lambda e^{\lambda t} = A\mathbf{v}e^{\lambda t} \Leftrightarrow \mathbf{v}\lambda = A\mathbf{v}$ or $A\mathbf{v} = \lambda\mathbf{v}$

$X = \mathbf{v}e^{\lambda t}$ is a solution of (*)

$$\Leftrightarrow$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

(i.e. λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector of A)

By knowing eigenvalues & eigenvectors we can find solutions of (*)

We will study solutions of homogeneous systems as

- distinct real eigenvalues of A {**this section**}
- distinct complex eigenvalues of A {**this section**}
- Repeated eigenvalues of A {**next section**}

Solving homogeneous linear systems

Case I: Distinct real eigenvalues

- Given $X' = AX$ (*) $(A: n \times n)$

and A has n real distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

- We need *n* independent solutions
- Each eigenvalue will give one.

❖ For each eigenvalue $\boxed{\lambda_i}$, find an associated eigenvector $\boxed{\mathbf{v}_i}$

➢ The solution vector associated to this eigenvalue is $\boxed{X_i = \mathbf{v}_i e^{\lambda_i t}}$

❖ Find all solution vectors and write general solution as

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

or $$X = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

Example 7.3.1 Solve $\begin{aligned} x_1' &= -4x_1 + 2x_2 \\ x_2' &= -\frac{5}{2}x_1 + 2x_2 \end{aligned}$.

Solution: Done in Class

Solving homogeneous linear systems

Case II: Distinct complex eigenvalues

- Given $X' = AX \qquad (*)$

  and A has distinct complex eigenvalues
- Recall: complex eigenvalues and eigenvectors occur in conjugate pairs.

❖ For each pair $\boxed{p \pm iq}$ of eigenvalues, find the associated pair of eigenvectors and split into real and imaginary parts as $\boxed{\mathbf{a} \pm i\mathbf{b}}$

> ➢ The solution vectors associated to the pair of eigenvalues $p \pm iq$ are

$$\boxed{\begin{aligned} X_1 &= e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right) \\ X_2 &= e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right) \end{aligned}}$$

❖ Find all solution vectors and write general solution.

Example 7.3.2 Solve $\begin{aligned} x_1' &= 5x_1 + x_2 \\ x_2' &= -2x_1 + 3x_2 \end{aligned}$.

Solution: Done in Class

Solving homogeneous linear systems
Mixture of Case I and Case II

- Given $X' = AX$ (*)

  and A has distinct real and complex eigenvalues

❖ For each real eigenvalue λ_i, find eigenvector $\mathbf{v}_i$

➢ Then $X_i = \mathbf{v}_i e^{\lambda_i t}$ is corresponding term in solution

❖ For each complex pair $p \pm iq$ of eigenvalues, find the pair of eigenvectors $\mathbf{a} \pm i\mathbf{b}$

➢ Then

$$e^{pt}\left(\mathbf{a}\cos qt - \mathbf{b}\sin qt\right)$$
$$e^{pt}\left(\mathbf{b}\cos qt + \mathbf{a}\sin qt\right)$$

are corresponding terms in the general solution

Example 7.3.3 Solve $X' = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} X$

Solution: Done in Class

Do Qs:1 – 25 *End of Section*