23. $c_0 = c_1 = 0$ and the recursion relation

$$(n^2 - n + 1)c_n + (n - 1)c_{n-1} = 0$$

for $n \geq 2$ imply that $c_n = 0$ for $n \geq 0$. Thus any assumed power series solution $y = \Sigma c_n x^n$ must reduce to the trivial solution $y(x) \equiv 0$.

25. This problem is pretty fully outlined in the textbook. The only hard part is squaring the power series:

$$\begin{aligned}&\left(1+c_3x^3+c_5x^5+c_7x^7+c_9x^9+c_{11}x^{11}+\cdots\right)^2\\ &\quad= x^2+2c_3x^4+\left(c_3^2+2c_5\right)x^6+\left(2c_3c_5+2c_7\right)x^8+\\ &\qquad\left(c_5^2+2c_3c_7+2c_9\right)x^{10}+\left(2c_5c_7+2c_3c_9+2c_{11}\right)x^{12}+\cdots\end{aligned}$$

SECTION 11.2

POWER SERIES SOLUTIONS

Instead of deriving in detail the recurrence relations and solution series for Problems 1 through 15, we indicate where some of these problems and answers originally came from. Each of the differential equations in Problems 1–10 is of the form

$$(Ax^2 + B)y'' + Cxy' + Dy = 0$$

with selected values of the constants A, B, C, D. When we substitute $y = \Sigma c_n x^n$, shift indices where appropriate, and collect coefficients, we get

$$\sum_{n=0}^{\infty}\left[An(n-1)c_n + B(n+1)(n+2)c_{n+2} + Cnc_n + Dc_n\right]x^n = 0.$$

Thus the recurrence relation is

$$c_{n+2} = -\frac{An^2+(C-A)n+D}{B(n+1)(n+2)}c_n \qquad \text{for } n \geq 0.$$

It yields a solution of the form

$$y = c_0\, y_{\text{even}} + c_1\, y_{\text{odd}}$$

where y_{even} and y_{odd} denote series with terms of even and odd degrees, respectively. The even-degree series $c_0 + c_2x^2 + c_4x^4 + \cdots$ converges (by the ratio test) provided that

$$\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_n x^n}\right| = \left|\frac{Ax^2}{B}\right| < 1.$$

Hence its radius of convergence is at least $\rho = \sqrt{|B/A|}$, as is that of the odd-degree series $c_1x + c_3x^3 + c_5x^4 + \cdots$. (See Problem 6 for an example in which the radius of convergence is, surprisingly, greater than $\sqrt{|B/A|}$.)

In Problems 1–15 we give first the recurrence relation and the radius of convergence, then the resulting power series solution.

1. $c_{n+2} = c_n;\qquad \rho = 1;\qquad c_0 = c_2 = c_4 = \cdots;\qquad c_1 = c_3 = c_4 = \cdots$

$$y(x) = c_0\sum_{n=0}^{\infty} x^{2n} + c_1\sum_{n=0}^{\infty} x^{2n+1} = \frac{c_0 + c_1 x}{1-x^2}$$

3. $c_{n+2} = -\dfrac{c_n}{(n+2)};\qquad \rho = \infty;$

$$c_{2n} = \frac{(-1)^n c_0}{(2n)(2n-2)\cdots\cdots 4\cdot 2} = \frac{(-1)^n c_0}{n!2^n};\quad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)(2n-1)\cdots\cdots 5\cdot 3} = \frac{(-1)^n c_1}{(2n+1)!!}$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!!}$$

5. $c_{n+2} = \dfrac{nc_n}{3(n+2)};\qquad \rho = 3;\qquad c_2 = c_4 = c_6 = \cdots = 0$

$$c_{2n+1} = \frac{2n-1}{3(2n+1)}\cdot\frac{2n-3}{3(2n-1)}\cdots\cdots\frac{3}{3(5)}\cdot\frac{1}{3(3)}c_1 = \frac{c_1}{(2n+1)3^n}$$

$$y(x) = c_0 + c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)3^n}$$

7. $c_{n+2} = -\dfrac{(n-4)^2}{3(n+1)(n+2)}c_n;\qquad \rho \geq \sqrt{3}$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0$, so y_{even} is a 4th-degree polynomial.

We find first that $c_3 = -c_1/2$ and $c_5 = c_1/120$, and then for $n \geq 3$ that

$$c_{2n+1} = \left(-\frac{(2n-5)^2}{3(2n)(2n+1)}\right)\left(-\frac{(2n-7)^2}{3(2n-2)(2n-1)}\right)\cdots\left(-\frac{1^2}{3(6)(7)}\right)c_5 =$$

$$= (-1)^{n-2}\frac{[(2n-5)!!]^2}{3^{n-2}(2n+1)(2n-1)\cdots 7\cdot 6}\cdot\frac{c_1}{120} = 9\cdot(-1)^n\frac{[(2n-5)!!]^2}{3^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+c_1\left[x-\frac{1}{2}x^3+\frac{1}{120}x^5+9\sum_{n=3}^{\infty}\frac{[(2n-5)!!]^2(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

9. $c_{n+2} = \dfrac{(n+3)(n+4)}{(n+1)(n+2)}c_n; \qquad \rho = 1$

$$c_{2n} = \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{(2n-1)(2n)}{(2n-3)(2n-2)}\cdots\frac{3\cdot 4}{1\cdot 2}c_0 = \frac{1}{2}(n+1)(2n+1)c_0$$

$$c_{2n+1} = \frac{(2n+2)(2n+3)}{(2n)(2n+1)}\cdot\frac{(2n)(2n+1)}{(2n-2)(2n-1)}\cdots\frac{4\cdot 5}{2\cdot 3}c_1 = \frac{1}{3}(n+1)(2n+3)c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

11. $c_{n+2} = \dfrac{2(n-5)}{5(n+1)(n+2)}c_n; \qquad \rho = \infty$

The factor $(n-5)$ yields $c_7 = c_9 = c_{11} = \cdots = 0$, so y_{odd} is a 5th-degree polynomial. We find first that $c_2 = -c_1$, $c_4 = c_0/10$ and $c_6 = c_0/750$, and then for $n \geq 4$ that

$$c_{2n} = \frac{2(2n-7)}{5(2n)(2n-1)}\cdot\frac{2(2n-5)}{5(2n-2)(2n-3)}\cdots\frac{2(1)}{5(8)(7)}c_6$$

$$= \frac{2^{n-3}(2n-7)!!}{5^{n-3}(2n)(2n-1)\cdots(8)(7)}\cdot\frac{c_0}{750} =$$

$$= \frac{5^3\cdot 6!}{2^3\cdot 750}\cdot\frac{2^n(2n-7)!!}{5^n(2n)(2n)\cdots(8)(7)\cdot 6!}\cdot c_1 = 15\cdot\frac{2^n(2n-7)!!}{5^n(2n)!}c_0$$

$$y(x) = c_1\left(x-\frac{4x^3}{15}+\frac{4x^5}{375}\right)+c_0\left[1-x^2+\frac{x^4}{10}+\frac{x^6}{750}+15\sum_{n=4}^{\infty}\frac{(2n-7)!!\,2^n}{(2n)!\,5^n}x^{2n}\right]$$

13. $c_{n+3} = -\dfrac{c_n}{n+3}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!\,3^n} + c_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{1\cdot 4\cdot\cdots\cdot(3n+1)}$$

15. $c_{n+4} = -\dfrac{c_n}{(n+3)(n+4)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_3 = 0$, so the recurrence relation yields $c_6 = c_{10} = \cdots = 0$ and $c_7 = c_{11} = \cdots = 0$ also. Then

$$c_{4n} = \frac{-1}{(4n)(4n-1)}\cdot\frac{-1}{(4n-4)(4n-5)}\cdot\cdots\cdot\frac{-1}{4\cdot 3}c_0 = \frac{(-1)^n c_0}{4^n n!\,(4n-1)(4n-5)\cdot\cdots\cdot 5\cdot 3},$$

$$c_{3n+1} = \frac{-1}{(4n+1)(4n)}\cdot\frac{-1}{(4n-3)(4n-4)}\cdot\cdots\cdot\frac{-1}{5\cdot 4}c_1 = \frac{(-1)^n c_1}{4^n n!\,(4n+1)(4n-3)\cdot\cdots\cdot 9\cdot 5}.$$

$$y(x) = c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n}}{4^n n!\,3\cdot 7\cdot\cdots\cdot(4n-1)}\right] + c_1\left[x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n+1}}{4^n n!\,5\cdot 9\cdot\cdots\cdot(4n+1)}\right]$$

17. The recurrence relation

$$c_{n+2} = -\frac{(n-2)c_n}{(n+1)(n+2)}$$

yields $c_2 = c_0 = y(0) = 1$ and $c_4 = c_6 = \cdots = 0$. Because $c_1 = y'(0) = 0$, it follows also that $c_1 = c_3 = c_5 = \cdots = 0$. Thus the desired particular solution is $y(x) = 1 + x^2$.

19. The substitution $t = x - 1$ yields $(1 - t^2)y'' - 6ty' - 4y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{n+4}{n+2}c_n.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = 1$, and therefore converges if $-1 < t < 1$. The initial conditions give $c_0 = 0$ and $c_1 = 1$, so $c_{\text{even}} = 0$ and

$$c_{2n+1} = \frac{2n+3}{2n+1}\cdot\frac{2n+1}{2n-1}\cdot\cdots\cdot\frac{7}{5}\cdot\frac{5}{3}c_1 = \frac{2n+3}{3}.$$

Thus

$$y = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)t^{2n+1} = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)(x-1)^{2n+1},$$

and the x-series converges if $0 < x < 2$.

21. The substitution $t = x+2$ yields $(4t^2+1)y'' = 8y$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{4(n-2)}{(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 1$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = 4$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2+4t^2 = 1+4(x+2)^2.$$

In Problems 23–26 we first derive the recurrence relation, and then calculate the solution series $y_1(x)$ with $c_0 = 1$ and $c_1 = 0$, the solution series $y_2(x)$ with $c_0 = 0$ and $c_1 = 1$.

23. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + \sum_{n=1}^{\infty}\left[c_{n-1} + c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{1}{2}c_0, \qquad c_{n+2} = -\frac{c_{n-1}+c_n}{(n+1)(n+2)} \text{ for } n \geq 1.$$

$$y_1(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots; \qquad y_2(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots$$

25. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty}\left[c_{n-2} + (n-1)c_{n-1} + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = c_3 = 0, \qquad c_{n+2} = -\frac{c_{n-2}+(n-1)c_{n-1}}{(n+1)(n+2)} \text{ for } n \geq 2.$$

$$y_1(x) = 1 - \frac{x^4}{12} + \frac{x^7}{126} + \frac{x^8}{672} + \cdots; \qquad y_2(x) = x - \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^7}{126} + \cdots$$

27. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty}\left[2c_{n-2} + (n+1)c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{c_0}{2}, \quad c_3 = -\frac{c_1}{3}, \qquad c_{n+2} = -\frac{2c_{n-2}+(n+1)c_n}{(n+1)(n+2)} \text{ for } n \geq 2.$$

With $c_0 = y(0) = 1$ and $c_1 = y'(0) = -1$, we obtain

$$y(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} - \frac{13x^7}{630} - \frac{143x^8}{40320} + \frac{31x^9}{22680} + \cdots.$$

Finally, $x = 0.5$ gives

$$\begin{aligned} y(0.5) &= 1 - 0.5 - 0.125 + 0.041667 - 0.002604 + 0.001042 \\ &\quad + 0.000629 - 0.000161 - 0.000014 + 0.000003 + \cdots \\ y(0.5) &\approx 0.415562 \approx 0.4156. \end{aligned}$$

29. When we substitute $y = \Sigma c_n x^n$ and $\cos x = \sum (-1)^n x^{2n}/(2n)!$ and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^6$, we obtain the equations

$$c_0 + 2c_2 = 0, \quad c_1 + 6c_3 = 0, \quad 12c_4 = 0, \quad -2c_3 + 20c_5 = 0,$$

$$\frac{1}{12}c_2 - 5c_4 + 30c_6 = 0, \quad \frac{1}{4}c_3 - 9c_5 + 42c_6 = 0,$$

$$-\frac{1}{360}c_2 + \frac{1}{2}c_4 - 14c_6 + 56c_8 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_8$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^6}{720} + \frac{13x^8}{40320} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{13x^7}{5040} + \cdots.$$

SECTION 11.3

FROBENIUS SERIES SOLUTIONS

1. Upon division of the given differential equation by x we see that $P(x) = 1 - x^2$ and $Q(x) = (\sin x)/x$. Because both are analytic at $x = 0$ — in particular, $(\sin x)/x \to 1$ as $x \to 0$ because

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

— it follows that $x = 0$ is an ordinary point.

3. When we rewrite the given equation in the standard form of Equation (3) in this section, we see that $p(x) = (\cos x)/x$ and $q(x) = x$. Because $(\cos x)/x \to \infty$ as $x \to 0$ it follows that $p(x)$ is not analytic, so $x = 0$ is an irregular singular point.

5. In the standard form of Equation (3) we have $p(x) = 2/(1 + x)$ and $q(x) = 3x^2/(1 + x)$. Both are analytic, so $x = 0$ is a regular singular point. The indicial equation is