$\boxed{\text{Ex } \frac{5}{245}}$ Solve $\quad y'' - (1+x)y = 0$

Sol. There are no singular points, therefore $\exists$ two power series solutions centered at 0, convergent for $|x| < \infty$.

Assume the solution is $y = \sum_{n=0}^{\infty} c_n x^n$, then

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substituting in the given DE, we have:

$$y'' - (1+x)y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - (1+x) \sum_{n=0}^{\infty} c_n x^n = 0$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1}$$

$\begin{cases} k = n-2 \\ n = k+2 \end{cases}$ $\qquad \begin{cases} k = n+1 \\ n = k-1 \end{cases}$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{n=0}^{\infty} c_n x^n - \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - c_0 - \sum_{k=1}^{\infty} c_k x^k - \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2C_2 - C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - C_k - C_{k-1} \right] x^k = 0$$

$$\Rightarrow 2C_2 - C_0 = 0 \quad \text{and} \quad (k+2)(k+1)C_{k+2} - C_k - C_{k-1} = 0$$

[using the identity property]

$$\Rightarrow C_2 = \tfrac{1}{2}C_0 \quad \text{and} \quad C_{k+2} = \frac{C_k + C_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \ldots$$

this is a three-term recurrence relation.

We can observe that the coefficients $C_3, C_4, C_5, \ldots$ are expressed in terms of both C_0 and C_1. To simplify our calculation, we first choose $C_0 \neq 0$ and $C_1 = 0$:

This will give coefficients for one solution expressed entirely in terms of C_0:

So, in this case, we have:

$$\left.\begin{aligned}
& C_2 = \tfrac{1}{2}C_0 \\
k=1 \Rightarrow \quad & C_3 = \frac{C_1 + C_0}{6} = \frac{C_0}{6} \\
k=2 \Rightarrow \quad & C_4 = \frac{C_2 + C_1}{12} = \frac{C_0}{24} \\
k=3 \Rightarrow \quad & C_5 = \frac{C_3 + C_2}{20} = \frac{\frac{C_0}{6} + \frac{C_0}{2}}{20} = \frac{C_0}{30} \\
& \vdots
\end{aligned}\right\} \quad (*)$$

and so on.

$$= 2C_2 - C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - C_k - C_{k-1} \right] x^k = 0$$

$$\Rightarrow 2C_2 - C_0 = 0 \quad \text{and} \quad (k+2)(k+1)C_{k+2} - C_k - C_{k-1} = 0$$

[using the identity property]

$$\Rightarrow C_2 = \tfrac{1}{2}C_0 \quad \text{and} \quad C_{k+2} = \frac{C_k + C_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \ldots$$

this is a three-term recurrence relation.

We can observe that the coefficients $C_3, C_4, C_5, \ldots$ are expressed in terms of both C_0 and C_1. To simplify our calculation, we first choose $C_0 \neq 0$ and $C_1 = 0$:

This will give coefficients for one solution expressed entirely in terms of C_0:

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k=3 \Rightarrow\quad & C_5 = \frac{C_3 + C_2}{20} = \frac{\frac{C_0}{6} + \frac{C_0}{2}}{20} = \frac{C_0}{30} \\
& \vdots
\end{aligned}\right\} (*)$$

and so on.

Next, choose $C_0 = 0$ and $C_1 \neq 0$:

This will give coefficients for the other solution expressed in terms of C_1:
So, in this case we have,

$$\left.\begin{aligned} C_2 &= \frac{C_0}{2} = 0 \\ K=1 \Rightarrow\ C_3 &= \frac{C_1+C_0}{6} = \frac{C_1}{6} \\ K=2 \Rightarrow\ C_4 &= \frac{C_2+C_1}{12} = \frac{C_1}{12} \\ K=3 \Rightarrow\ C_5 &= \frac{C_3+C_2}{120} = \frac{C_1}{120} \\ &\ \vdots \end{aligned}\right\} \quad (**)$$

and so on.

Now, remember our solution is $y = \sum_{n=0}^{\infty} C_n x^n$

So, the coefficients in $(*)$ will give the following solution:

$$y_1(x) = C_0 + 0\cdot x + \frac{C_0}{2}x^2 + \frac{C_0}{6}x^3 + \frac{C_0}{24}x^4 + \frac{C_0}{30}x^5 + \cdots$$

and the coefficients in $(**)$ will give the second solution:

$$y_2(x) = 0 + C_1 x + 0\cdot x^2 + \frac{C_1}{6}x^3 + \frac{C_1}{12}x^4 + \frac{C_1}{120}x^5 + \cdots$$

The general solution of the given DE is:

$$y = K_1 y_1(x) + K_2 y_2(x)$$