Example 3 [Section 6.1.2]

Solve the IVP

$$(x-1)y''-xy'+y=0 \qquad (*)$$

$$y(0)=-2, \quad y'(0)=6$$

Solution

- Choosing ordinary point x_0

  - We choose $x_0=0$ [since initial condition is at $x=0$]

- Considering series solution y and putting y, y', y'' in (*)

  - Take $y=\sum_{n=0}^{\infty} c_n x^n$.
  - Then $y'=\sum_{n=1}^{\infty} nc_n x^{n-1}$, $y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$.
  - Putting in (*) gives

$$(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

- Simplifying & shifting index

  - Above equation can be written as

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{\text{Put } k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\text{Put } k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{\text{Put } k=n} + \overbrace{\sum_{n=0}^{\infty} c_n x^n}^{\text{Put } k=n} = 0$$

$$\Rightarrow \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

Writing each series in the form involving x^k

- Writing as single series

  - Above equation implies

$$-2c_2 + c_0 + \sum_{k=1}^{\infty}\left[-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k\right]x^k = 0$$

• Comparing coefficients of powers of x

- Comparing coefficients of powers of x gives

  $-2c_2 + c_0 = 0$

  $-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k = 0$ for $k \geq 1$

- Which gives

  $c_2 = \dfrac{c_0}{2}$

  $c_{k+2} = \dfrac{k}{k+2}c_{k+1} - \dfrac{(k-1)}{(k+2)(k+1)}c_k$ for $k \geq 1$

Recurrence relation to determine coefficients c_n

• Finding coefficients c_n 's

- $k = 1 \Rightarrow \quad c_3 = \dfrac{1}{3}c_2 - 0c_1 = \dfrac{c_0}{6}$
- $k = 2 \Rightarrow \quad c_4 = \dfrac{2}{4}c_3 - \dfrac{1}{4\cdot 3}c_1 = \dfrac{1}{2}\cdot\dfrac{c_0}{6} - \dfrac{1}{12}\cdot\dfrac{c_0}{2} = \dfrac{c_0}{24}$
- $\vdots$

• Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\cdots$$

$$= c_1 x + c_0 + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \cdots\cdots$$

Using above

$$= c_1 x + c_0\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\cdots\right)$$

General solution

• Writing solution of IVP

- From above we have $y' = c_1 + c_0\left(\dfrac{2x}{2} + \dfrac{3x^2}{6} + \dfrac{4x^3}{24} + \cdots\cdots\right)$
- Hence $\quad y(0) = -2 \quad \Rightarrow \quad c_0 = -2$

  and $\quad y'(0) = 6 \quad \Rightarrow \quad c_1 = 6$
- Therefore solution of IVP is

$$y = 6x - 2\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\cdots\right).$$

- Writing solution in better form

May not be possible always

- We have found the solution

$$y = 6x - 2\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\cdots\right)$$

Is this a well known power series of some function?

- We can write as

$$y = 6x - 2\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\right)$$

- Which implies

$$y = 6x - 2\left(e^x\right)$$

- Hence the solution of IVP is

$$y = 6x - 2e^x$$

Example 4 [Section 6.1.2]

Solve $y'' + \sin(x)y = 0 \qquad (*)$

about the ordinary point $x_0 = 0$.

Solution

- Considering series solution y and putting y, y', y'' in (*)

  - Take $y = \sum_{n=0}^{\infty} c_n x^n$.
  - Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.
  - Putting in (*) gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (\sin x)\sum_{n=0}^{\infty} c_n x^n = 0$$

To proceed we need to write $\sin x$ as a power series

- Putting series expansion of $\sin x$ and simplifying

  - Using series expansion of $\sin x$ we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(x - \frac{x^3}{3!} + \cdots\right)\sum_{n=0}^{\infty} c_n x^n = 0$$

  - This implies

$$\left[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots\right] +$$
$$+\left(x - \frac{x^3}{3!} + \cdots\right)\left[c_0 + c_1 x + c_2 x^2 + \cdots\right] = 0$$

How to handle such products?

See below.

$$\left(x - \frac{x^3}{3!} + \cdots\right)\left[c_0 + c_1 x + c_2 x^2 + \cdots\right] = \left(c_0 x + c_1 x^2 + c_2 x^3 \cdots\right) + \left(-c_0\frac{x^3}{3!} - c_1\frac{x^4}{3!} - c_2\frac{x^5}{3!}\cdots\right) + \cdots$$
$$= c_0 x + c_1 x^2 + \left(c_2 - \frac{c_0}{6}\right)x^3 + \cdots$$

  - Hence series becomes

$$\left[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots\right] + \left[c_0 x + c_1 x^2 + \left(c_2 - \frac{c_0}{6}\right)x^3 + \cdots\right] = 0$$

$$\Rightarrow 2c_2 + \left(6c_3 + c_0\right)x + \left(12c_4 + c_1\right)x^2 + \left(20c_5 + c_2 - \frac{c_0}{6}\right)x^3 + \cdots = 0 \qquad (**)$$

• Comparing coefficients of powers of x to get c_n's

- Comparing coefficients of powers of x in (**) gives

$$c_2 = 0$$

$$c_3 = -\frac{c_0}{6}$$

$$c_4 = -\frac{c_1}{12}$$

$$c_5 = -\frac{c_2}{20} + \frac{c_0}{120} = \frac{c_0}{120}$$

$$\vdots$$

• Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \cdots\cdots$$

$$= \left(c_0 - c_0 \frac{x^3}{6} + c_0 \frac{x^5}{120} + \cdots\cdots \right) + \left(c_1 x - c_1 \frac{x^4}{12} + \cdots\cdots \right)$$

$$= c_0 \left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\cdots \right) + c_1 \left(x - \frac{x^4}{12} + \cdots\cdots \right)$$

General solution