KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 550-(081)

Exam I

Time: 90 Minutes

Name: Solution Sec.# I.D. #

Show All Necessary Work

Question	Points
1	7
2	10
3	[illegible]
4	7
5	[illegible]
Total	

1. Let V be a vector space and let W_1, W_2 be finite dimensional subspaces of V. Prove that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

See your notes

2. Let $T: R^3 \to R^2$ given by

$$T(x,y,z) = (2x+y, x+y-z)$$

Let S and B be the standard bases for R^3 and R^2 respectively, and let $S' = \{(2,0,0),(0,-1,0),(0,0,-2)\}$ and $B' = \{(1,1),(1,-1)\}$ be some ordered bases for R^3 and R^2 respectively.

(a) Show that T is a linear transformation.

Let $\alpha, \beta \in R^3$, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$. Then

$T(\alpha+\beta) = T[(x_1+x_2, y_1+y_2, z_1+z_2)] = \left(2(x_1+x_2)+y_1+y_2,\ x_1+x_2+y_1+y_2-z_1-z_2\right)$

$= \left(2x_1+2x_2+y_1+y_2,\ x_1+y_1-z_1+x_2+y_2-z_2\right)$

$= (2x_1+y_1, x_1+y_1-z_1) + (2x_2+y_2, x_2+y_2-z_2) = T(\alpha)+T(\beta)$

Also, if $c \in R$,

$T(c\alpha) = T(cx_1, cy_1, cz_1) = \left(2cx_1+cy_1,\ cx_1+cy_1-cz_1\right) = c\left(2x_1+y_1, x_1+y_1-z_1\right)$

(b) Is T singular, is it onto?

$= cT(\alpha)$.

Suppose $T(\alpha) = 0$. $\Rightarrow T(x,y,z) = 0 = (2x+y, x+y-z) = (0,0)$

$\Rightarrow \begin{cases} 2x+y=0 \\ x+y-z=0 \end{cases}$ This system has a non-trivial solution i.e. $\alpha \neq 0$. for example $\alpha = (1,-2,-1)$ $T(\alpha) = (0,0)$

Hence T is non singular.

Let $(x,y) \in R^2$. Check if $\exists\ (a,b,c) \in R^3 \ni T(a,b,c) = (x,y)$.

$\Rightarrow T(a,b,c) = (2a+b, a+b-c) = (x,y) \Rightarrow$ Solving for a, b, c, we get

$a = x-y$, $b = 2y-x$, $c=0$ $\Rightarrow T(x-y, 2y-x, 0) = (x,y) \Rightarrow T$ is onto.

(c) What is the nullity of T?

Nullity of T + Rank $T = \dim R^3 = 3$

By (b) above, T is onto $\Rightarrow$ range $T = R^2 \Rightarrow$ rank $T = \dim R^2 = 2$

Nullity of $T = 3-2 = 1$

(d) What is the rank of T?

From (c), rank $T = 2$

(e) Find the transition matrix P from B' to B.

Recall, $[\alpha]_{B} = \underset{B \leftarrow B'}{P} [\alpha]_{B'}$, where $P_j = [\beta'_j]_B$

$[\beta'_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[\beta'_2]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$S = \{\alpha_1, \alpha_2, \alpha_3\}$
$= \{e_1, e_2, e_3\} = \{(1,0,0), \ldots$

$S' = \{\alpha'_1, \alpha'_2, \alpha'_3\}$
$= \{(2,0,0), (0,-1,0), (0,0,-2)\}$

$B = \{\beta_1, \beta_2\}$
$= \{e_1, e_2\} = \{(1,0), (0,1)\}$

$B' = \{\beta'_1, \beta'_2\}$
$= \{(1,1), (1,-1)\}$

(f) Find the transition matrix Q from S' to S.

$[\alpha]_S = \underset{S \leftarrow S'}{Q} [\alpha]_{S'}$, where $Q_j = [\alpha'_j]_S$

$[\alpha'_1]_S = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$[\alpha'_2]_S = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$[\alpha'_3]_S = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

3. (a) Let S be a linearly independent subset of a vector space V. Let β be a vector in V such that $\beta \notin span(S)$. Show that $S \cup \{\beta\}$ is linearly independent.

See your notes

(b) Show that if V is an n-dimensional vector space, then any linearly independent subset S of V is a part of a basis for V.

See your notes

4. Prove of disprove each of the following statements:

(a) If S is an infinite subset of an n-dimensional vector space V, then space S is not finite dimensional.

Consider the vector space $V = \mathbb{R}^n$. So $\dim V = n$.

Let $S = \{(x, 0, 0, 0, \dots, 0) : x \in \mathbb{R}\}$. Then

S is an infinite subset of V but span S is of

dimension one. Thus the statement is false.

(b) There is exactly one linear transformation $T : R^2 \to R^2$ such that $T(1,3) = (1,0)$ and $T(3,1) = (0,1)$.

This statement is true if $(1,3), (3,1)$ are linearly indep.

$c_1(1,5) + c_2(5,1) = (0,0) \Rightarrow c_1 = c_2 = 0$

$\Rightarrow (1,5), (5,1)$ are linearly independent and hence

they form a basis for $\mathbb{R}^2$. Thus, there is exactly

one linear transformation T such that $T(1,3) = (1,0)$ and

$T(3,1) = (0,1)$

5. Let T be a linear operator on an n-dimensional vector space V such that the range and null space of T are identical. Show that n is even.

range(T) = nullspace of T

$\Rightarrow$ rank(T) = nullity(T)

Now, nullity(T) + rank(T) = dim V = n

$\Rightarrow$ rank(T) + rank(T) = n

2 rank(T) = n

Hence T is even.