Applications of First Order Differential Equations - (1.3)(2.7)(2.8)

1. **Growth and Decay:**

Consider the initial value problem:

$$\frac{dP}{dt} = kP, \quad P(0) = P_0.$$

Function $P(t)$ represents population at the time t. When $k > 0$, the population is increasing and when $k < 0$, the population is decreasing. The equation is separable and solution is

$$P(t) = P_0 e^{kt}.$$

Logistic Equation:

$$\frac{dP}{dt} = P(a - bP), \quad P(0) = P_0. \quad \left(\frac{1}{P}\frac{dP}{dt} = (a - bP) \text{ or } \frac{dP}{dt} = aP - bP^2\right)$$

Logistic curves have proved to be quite accurate in predicting the growth patterns, in a limited space, of certain types of bacteria, water fleas, and fruit flies. The equation is separable, and also a Bernoulli equation. The solution is:

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

Example *Suppose a student carrying a flu virus returns to an isolated college campus of* 1000 *students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after* 6 *days if it is further observed that after* 4 *days* 50 *students are infected.*

The mathematical model is:

$$\frac{dx}{dt} = kx(1000 - x), \; x(0) = 1.$$

Since

$$\frac{dx}{dt} = 1000kx - kx^2, \quad a = 1000k, \; b = k, \; P_0 = 1.$$

Hence

$$x(t) = \frac{1000k}{k + (1000k - k)e^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}$$

It is known that $x(4) = 50$. Then solve k from the equation:

$$50 = \frac{1000}{1 + 999e^{-1000k(4)}} \Rightarrow k = -\frac{1}{4000}\ln\frac{19}{999}$$

$$x(t) = \frac{1000}{1 + 999e^{0.25\ln(19/999)\,t}}, \quad x(6) = \frac{1000}{1 + 999e^{0.25\ln(19/999)(6)}} = 276$$

About 276 students are infected after 6 days.

2. **Newton's Law of Cooling:**

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0.$$

Function $T(t)$ represents temperature of an object at the time t where T_0 is the initial temperature of the object and T_m is the temperature surrounded the object. When $k > 0$, T is increasing and when $k < 0$, T is decreasing. The equation is separable and the solution is:

$$T(t) = T_m + e^{kt}(T_0 - T_m).$$

Example *A small metal bar, whose initial temperature was* $20^o C$, *is dropped into a container of boiling water. How long will it take the bar to reach* $90^o C$ *if is known that its temperature increased* $2^o C$ *in* 1 *second? How long will it take the bar to reach* $98^o C$?

Let T be the temperature of the bar at the t. Then we know: $T_m = 100^o C$, $T_0 = 20^o C$. So,

$$T(t) = 100 + (20 - 100)e^{kt} = 100 - 80e^{kt}.$$

Since $T(1) = 2 + 20 = 22^oC$, solve k from the equation:

$$22 = 100 - 80e^{k}.$$

$$k = \ln\frac{39}{40} \text{ and } T(t) = 100 - 80e^{\ln(39/40)\,t}.$$

Find t when $T(t) = 90^oC$.

$$90 = 100 - 80e^{\ln(39/40)\,t} \Rightarrow t = -\frac{\ln 8}{\ln\frac{39}{40}} = 82.13 \text{ seconds}$$

Find t when $T(t) = 98^oC$.

$$98 = 100 - 80e^{\ln(39/40)\,t} \Rightarrow t = -\frac{\ln 40}{\ln\frac{39}{40}} = 145.70 \text{ seconds}$$

3. LR, and RC Circuits:

$$L\frac{dI}{dt} + RI = E(t), \quad I(0) = I_0$$

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \; Q(0) = Q_0$$

Functions $I(t)$ and $Q(t)$ are current and charge at the time t, respectively. L, R, and C are constants for inductor, resistor and capacitor, respectively. Function $E(t)$ is voltage on the circuit at the time t. By Kirchhoff's second law,

$$E_R = RI, \quad E_L = L\frac{dI}{dt}, \quad E_C = \frac{1}{C}\int I(t)dt, \quad \frac{dQ}{dt} = I$$

These two equations are linear (in I and Q). The solutions depend on given $E(t)$.

Example *Consider a LR-circuit with* $L = 4$, $R = 2$, $I(0) = 0$, *and* $E(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1 \end{cases}$.

Solve $I(t)$.

Solve $I(t)$ from the initial value problem:

$$4\frac{dI}{dt} + 2I = E(t), \quad I(0) = 0 \Rightarrow \frac{dI}{dt} + \frac{1}{2}I = \frac{1}{4}E(t), \quad I(0) = 0$$

It is linear in I.

a. $h(t) = \int \frac{1}{2}dt = \frac{1}{2}t$, I.F.: $e^{t/2}$

b. $\begin{cases} k(t) = \int e^{t/2}(t)dt = 2e^{\frac{1}{2}t}t - 4e^{\frac{1}{2}t}, \; 0 \le t \le 1 \\ k(t) = \int e^{t/2}(1)dt = 2e^{\frac{1}{2}t}, \; t > 1 \end{cases}$

c. $\begin{cases} I(t) = e^{-t/2}(C_1 + 2te^{t/2} - 4e^{t/2}), \; 0 \le t \le 1 \\ I(t) = e^{-t/2}(C_2 + 2e^{t/2}), \; t > 1 \end{cases}$

d. When $t = 0$, $I(0) = 0$. Solve C_1 :

$$0 = e^0(C_1 + 2e^0(0) - 4e^0), \; C_1 = 4.$$

$$I(t) = e^{-t/2}(4 + 2e^{t/2}t - 4e^{t/2}), \; 0 \le t \le 1$$

In particular, $I(1) = e^{-1/2}(4 + 2e^{1/2} - 4e^{1/2}) = 4e^{-1/2} + 2 - 4 = 4e^{-1/2} - 2$.

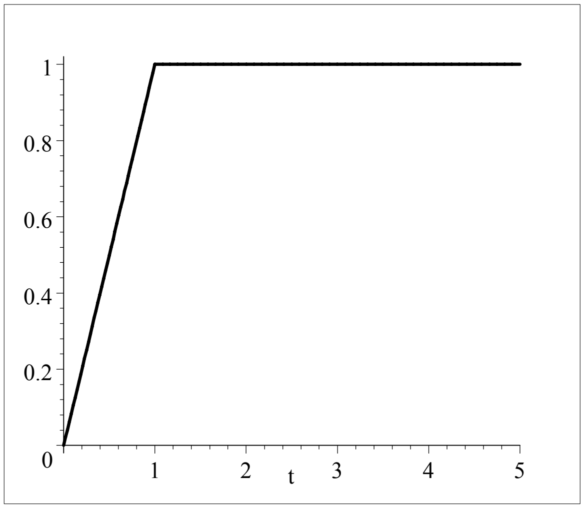
When $t = 1$, $I(1) = 4e^{-1/2} - 2$. Solve C_2 :

$$4e^{-1/2} - 2 = e^{-1/2}(C_2 + 2e^{1/2}) \Rightarrow 4 - 2e^{1/2} = C_2 + 2e^{1/2}, \; C_2 = 4 - 4e^{1/2}$$

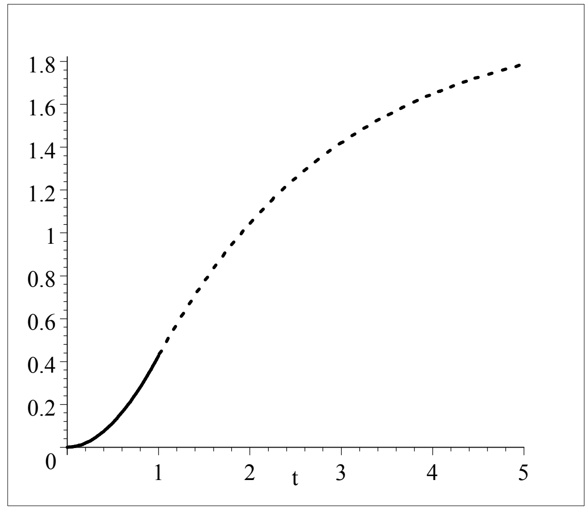
$$I(t) = e^{-t/2}(4 - 4e^{1/2} + 2e^{t/2}), \; t > 1$$

Solution:

$$I(t) = \begin{cases} e^{-t/2}(4 + 2te^{t/2} - 4e^{t/2}), \ 0 \le t \le 1 \\ e^{-t/2}(4 - 4e^{1/2} + 2e^{t/2}), \ t > 1 \end{cases} .$$



$E(t)$



$I(t)$