Ch-8 Systems of linear 1st order differential equations

Section 8.1 *Preliminary theory*

Until now, while studying differential equations, we only considered situations where only one quantity (or variable) was changing [or only one uknown function and one equation was involved]. e.g. $\frac{d^2x}{dt^2}+7\frac{dx}{dt}+x=\cos t$.

Here we will study systems of differential equations where more than one differential equation and the change in more than one quantity (or variable) is involved.

Learning outcomes

After completing this section, you will inshaAllah be able to

1. explain what is a first order linear system
2. learn how to write a system of 1st order linear differential equations in matrix form
3. tell basic facts about solutions of homogeneous linear systems
4. check linear independence of solutions of homogeneous systems

Linear systems of 1st order differential equations
(Normal Form)

- **1st order linear system:** A system of 1st order ODE's in which unknowns appear linearly (i.e. linear in dependent variables)

- **Examples:**

1.
$$\frac{dx_1}{dt} = 3x_1 - 5x_2 + 2e^t$$
$$\frac{dx_2}{dt} = 2x_1 + x_2 - \sin t$$
Linear

2.
$$\frac{dx_1}{dt} = 3x_1x_2 - 5x_2$$
$$\frac{dx_2}{dt} = 2x_1 + x_2$$
Non Linear

3.
$$\frac{dx_1}{dt} = -2x_2$$
$$\frac{dx_2}{dt} = 3(x_1)^2 + 4$$
Non Linear

Normal form of a 1st order linear system

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots\cdots + a_{1n}(t)x_n + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots\cdots + a_{2n}(t)x_n + f_2(t)$$
$$\vdots$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots\cdots + a_{nn}(t)x_n + f_n(t)$$

(*)

- If all $f_i(t) = 0$ then homogeneous system
- Otherwise non-homogeneous

Matrix form of a linear system

- The above linear system (*) can be written in matrix form as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

- If we set $X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

then we have

$$\frac{dX}{dt} = A(t) \cdot X + F$$

or

$$X' = AX + F \qquad (**)$$

Example 1 Write $\begin{aligned} x_1' &= 3x_1 - 5x_2 + 2e^t - t \\ x_2' &= 2x_1 + x_2 - \sin t \end{aligned}$ in matrix form

Solution: Done in Class

Example 2 Write the system $X' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} X$ in the normal form

Solution: Done in Class

Meaning of solution of a homogeneous system

- Given a homogeneous linear system

$$X' = AX \qquad (*) \quad \text{where } X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Solution in the form of a vector

- Solution: Functions $x_1, x_2, \cdots, x_n$ or a vector $X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ which satisfies equation (*).

Example 3 Consider the system $\begin{aligned} x_1' &= -4x_1 + 2x_2 \\ x_2' &= -\frac{5}{2}x_1 + 2x_2 \end{aligned}$ which in matrix form is given as $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $X' = AX$.

Show that $X_1 = \begin{bmatrix} 2e^t \\ 5e^t \end{bmatrix}$ and $X_2 = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ are two solutions of the system.

Is $c_1X_1 + c_2X_2$ also a solution of the system?

Solution: Done in class

Facts about solutions of homogeneous linear systems

Given a homogeneous linear system of *n* equations

$$X' = AX \qquad (*)$$

If vectors $X_1, X_2, \cdots, X_k$ are solutions of (*) then $c_1X_1 + c_2X_2 + \cdots + c_kX_k$ is also a solution

- Linear combination of solutions is a solution
- Can generate infinitely many solutions from few known solutions

How many solutions do we need?

(to know all solutions)

If $X_1, X_2, \cdots, X_n$ are linearly independent solutions of (*) then the general solution of (*) is

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

- All solutions can be obtained from *n* independent solutions
- How to check independence {see next}

So we need to know *n* independent solutions

How to check linear independence of solutions?

Given a homogeneous linear system of n equations

$$X' = AX \qquad (*)$$

and solution vectors $X_1, X_2, \cdots, X_n$

The solutions $X_1, X_2, \cdots, X_n$ are linearly independent

$$\Leftrightarrow$$

$$W(X_1, X_2, \cdots, X_n) = \begin{vmatrix} X_1 & X_2 & \cdots & X_n \end{vmatrix} \neq 0$$

for one value of t

The function W is called Wronskian of solution vectors

Example 4 Show that $X_1 = \begin{bmatrix} 2e^t \\ 5e^t \end{bmatrix}$, $X_2 = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ are linearly independent solutions of $X' = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix} X$.

Solution: Done in class

Exercise Show that $X_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, X_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, X_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$ are linearly independent solutions of

$$X' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} X.$$

Wrap Up
(General solution of homogeneous system)

- Given a homogeneous system $X' = AX$ (*)

  (where A is $n \times n$ matrix)

Fundamental set of solutions

is

a set of n linearly independent solutions

$X_1, X_2, \cdots, X_n$

If $X_1, X_2, \cdots, X_n$ is a fundamental solution

then

the general solution is

$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$

Exercise: Consider the system $X' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} X$

a) Show that $X_1 = \begin{bmatrix} \cos t \\ -\frac{\cos t}{2} + \frac{\sin t}{2} \\ -\cos t - \sin t \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} \sin t \\ -\frac{\sin t}{2} - \frac{\cos t}{2} \\ -\sin t + \cos t \end{bmatrix}$

form a fundamental set of solutions of the system.

b) Write the general solution of the system.

End of 8.1