SECTION 3.2

MATRICES AND GAUSSIAN ELIMINATION

Because the linear systems in Problems 1-10 are already in echelon form, we need only start at the end of the list of unknowns and work backwards.

1. Starting with $x_3 = 2$ from the third equation, the second equation gives $x_2 = 0,$ and then the first equation gives $x_1 = 1.$

3. If we set $x_3 = t$ then the second equation gives $x_2 = 2 + 5t,$ and next the first equation gives $x_1 = 13 + 11t.$

5. If we set $x_4 = t$ then the third equation gives $x_3 = 5 + 3t,$ next the second equation gives $x_2 = 6 + t,$ and finally the first equation gives $x_1 = 13 + 4t.$

7. If we set $x_3 = s$ and $x_4 = t,$ then the second equation gives $x_2 = 7 + 2s - 7t,$ and next the first equation gives $x_1 = 3 - 8s + 19t.$

9. Starting with $x_4 = 6$ from the fourth equation, the third equation gives $x_3 = -5,$ next the second equation gives $x_2 = 3,$ and finally the first equation gives $x_1 = 1.$

In each of Problems 11-22, we give just the first two or three steps in the reduction. Then we display a resulting echelon form **E** of the augmented coefficient matrix **A** of the given linear system, and finally list the resulting solution (if any). The student should understand that the echelon matrix **E** is not unique, so a different sequence of elementary row operations may produce a different echelon matrix.

11. Begin by interchanging rows 1 and 2 of **A**. Then subtract twice row 1 both from row 2 and from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}; \quad x_1 = 3,\ x_2 = -2,\ x_3 = 4$$

13. Begin by subtracting twice row 1 of **A** both from row 2 and from row 3. Then add row 2 to row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 3 & 3 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 4 + 3t,\ x_2 = 3 - 2t,\ x_3 = t$$

15. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and five times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \text{The system has no solution.}$$

17.

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 3x_4 &= 4 \\ 2x_1 - 6x_2 - 5x_3 - 5x_4 &= 5 \\ 3x_1 - \ \ x_2 - 4x_3 - 5x_4 &= -7 \end{aligned}$$

[The first printing of the textbook had a misprinted 2 (instead of 4) as the right-hand side constant in the first equation.] Begin by subtracting twice row 1 from row 2 of **A**, and three times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -4 & -3 & -3 & 4 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; \quad x_1 = 3 - 2t,\ x_2 = -4 + t,\ x_3 = 5 - 3t,\ x_4 = t$$

19. Begin by interchanging rows 1 and 2 of **A**. Then subtract three times row 1 from row 2, and four times row 1 from row 3.

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & 5 & -5 & -7 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad x_1 = 3 - s - t,\ x_2 = 5 + 2s - 3t,\ x_3 = s,\ x_4 = t$$

21. Begin by subtracting twice row 1 from row 2, three times row 1 from row 3, and four times row 1 from row 4.

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 5 & 1 & 20 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}; \quad x_1 = 2,\ x_2 = 1,\ x_3 = 3,\ x_4 = 4$$

23. If we subtract twice the first row from the second row, we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & k-2 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has no solutions unless $k = 2$, in which case it has infinitely many solutions given by $x_1 = \frac{1}{3}(1-2t)$, $x_2 = t$.

25. If we subtract twice the first row from the second row, we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 11 \\ 0 & k-4 & -1 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has a unique solution if $k \neq 4$, but no solution if $k = 4$.

27. If we first subtract twice the first row from the second row, then subtract 4 times the first row from the third row, and finally subtract the second row from the third row , we obtain the echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 0 & k-11 \end{bmatrix}$$

of the augmented coefficient matrix. It follows that the given system has no solution unless $k = 11$, in which case it has infinitely many solutions with x_3 arbitrary.

29. In each of parts (a)-(c), we start with a typical 2×2 matrix $\mathbf{A}$ and carry out two row successive operations as indicated, observing that we wind up with the original matrix $\mathbf{A}$.

(a) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{cR2} \begin{bmatrix} s & t \\ cu & cv \end{bmatrix} \xrightarrow{(1/c)R2} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

(b) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} u & v \\ s & t \end{bmatrix} \xrightarrow{SWAP(R1,R2)} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

(c) $$\mathbf{A} = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \xrightarrow{cR1+R2} \begin{bmatrix} u & v \\ cu+s & cv+t \end{bmatrix} \xrightarrow{(-c)R1+R2} \begin{bmatrix} s & t \\ u & v \end{bmatrix} = \mathbf{A}$$

Since we therefore can "reverse" any single elementary row operation, it follows that we can reverse any finite sequence of such operations — on at a time — so part (d) follows.