KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 202-(072)

Exam I

March 18, 2008

Time: 80 Minutes

Name: Solution Sec.# I.D. # Ser. #

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points
1	12
2	6
3	12
4	6
5	12
6	13
7	13
8	13
9	13
Total	100

12 **1.** Consider the autonomous first order differential equation $\frac{dy}{dx} = 15 - 2y - y^2$.

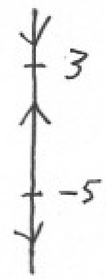
a) Find the critical points and classify each one in terms of its stability.

$f(y) = 0 \Rightarrow 15 - 2y - y^2 = 0$

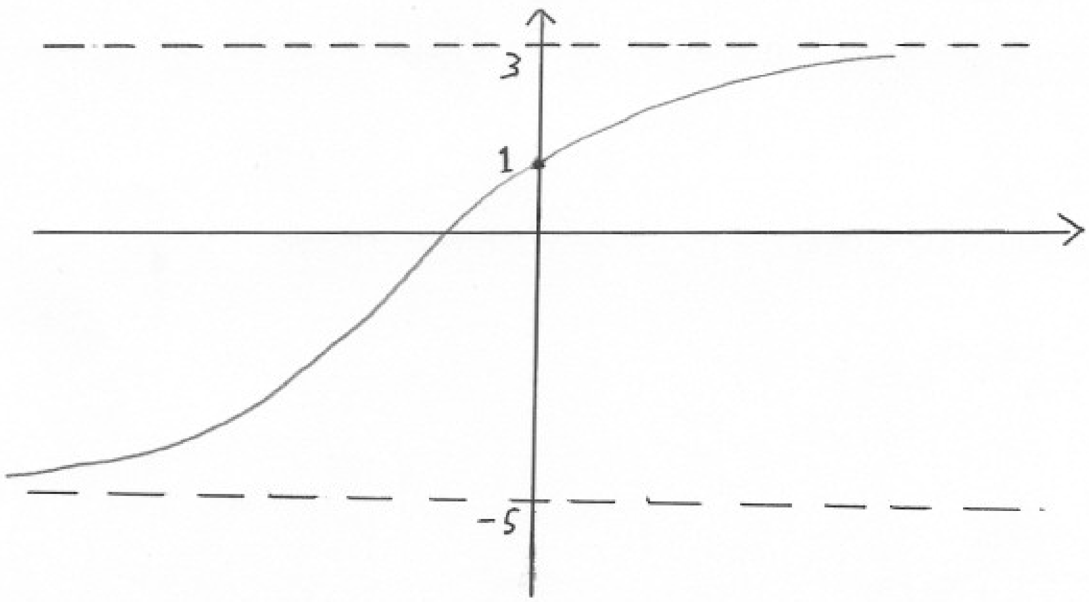
$(5+y)(3-y) = 0$

Critical pts: $y = -5, 3$

unstable   asym. stable



b) Sketch the graph of the solution curve determined by the initial condition $y(0) = 1$.



6 **2.** It is known that $y = \dfrac{2 + 2ce^{4x}}{1 - ce^{4x}}$ is a one parameter family of Solutions of the ODE $y' = y^2 - 4$.
Find a Singular Solution of this ODE.

$y' = y^2 - 4 = (y-2)(y+2) \Rightarrow y = 2, -2$ are solutions.

Note that $y = -2$ is a solution that can not be obtained from the above family of solutions. So, $y = -2$ is a singular solution.

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3. Write the 1[st] Order Differential Equations in the **Standard Form** in **one of the following Types: Separable, Linear in** *y* (or **in** *x*), **Homogeneous (with its degree), Bernoulli**, or **Exact**.

No	Equation	Work out Standard Form	Type
i	$[3x^2y^2+\sin y]\frac{dy}{dx}=-(2xy^3+e^x)$	$(3x^2y^2+\sin y)dy=-(2xy^3+e^x)dx$ $(2xy^3+e^x)dx+(3x^2y^2+\sin y)dy=0$ $M\,dx+N\,dy=0$ $\frac{\partial M}{\partial y}=6xy^2=\frac{\partial N}{\partial x}$	Exact
ii	$[x^2-1]dy=[xy-y+1]dx$	$\frac{dy}{dx}=\frac{xy-y+1}{x^2-1}$ $\frac{dy}{dx}-\left(\frac{x-1}{x^2-1}\right)y=\frac{1}{1-x^2}$ $\frac{dy}{dx}+P(x)y=f(x)$	Linear
iii	$\frac{dy}{dx}=1+x^2+y^2+x^2y^2$	$\frac{dy}{dx}=x^2(1+y^2)+1+y^2$ $=(1+y^2)(x^2+1)$ $\frac{dy}{1+y^2}=(x^2+1)dx$ or $\frac{dy}{dx}=f(y)\,g(x)$	Separable
v	$(4y-x^4y^2)dx=-xdy$	$\frac{dy}{dx}+\frac{4y-x^4y^2}{x}=0$ $\frac{dy}{dx}+\frac{4}{x}y=x^3y^2$ $\frac{dy}{dx}+P(x)y=f(x)y^n$	Bernoulli

6 **4.** Determine How many solutions are there to the initial value problem

$$\frac{1}{x^2}\frac{dy}{dx} + y^2 = \frac{1}{x}, \qquad y(0) = 2.$$ **Justify** your answer.

$$\frac{dy}{dx} = x - x^2y^2 = f(x,y)$$

$f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous $\Rightarrow \exists$ a unique solution

12 **5.** Verify that the following DE is not exact and then find an **integrating factor** that can be used in order to convert it to exact: **(Do not solve the equation)**

$$2y(x^2 - y + x)dx + (x^2 - 2y)dy = 0$$

$$\underbrace{(2x^2y - 2y^2 + 2xy)}_{M}dx + \underbrace{(x^2 - 2y)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = 2x^2 - 4y + 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Not exact.}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x^2 - 4y$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x^2 - 2y}(2x^2 - 4y) = \frac{2\cancel{(x^2-2y)}}{\cancel{x^2-2y}} = 2$$

$$\mu = e^{\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]dx} = e^{\int 2dx} = e^{2x}$$

13 **6.** Solve: $\dfrac{dx}{dt}+\dfrac{x}{1+t}=e^t$

$$\frac{dx}{dt}+\left(\frac{1}{1+t}\right)x=e^t$$

of the form: $\frac{dx}{dt}+P(t)x=g(t) \Rightarrow$ Linear

$$\mu=e^{\int \frac{1}{1+t}dt}=e^{\ln|1+t|}=1+t$$

to find $\int te^t dt$:

put $u=t$, $dv=e^t dt$
$du=1$, $v=e^t$

$\Rightarrow \int te^t dt = te^t - \int e^t dt$
$= te^t - e^t$

$$\frac{d}{dt}\left(x(1+t)\right)=(1+t)e^t$$

$$x(1+t)=\int(1+t)e^t dt=\int e^t dt+\int te^t dt$$

$$=\cancel{e^t}+te^t-\cancel{e^t}+C=te^t+C$$

The solution is $\boxed{x=\dfrac{te^t+C}{1+t}}$

13 **7.** Solve: $(e^x+e^{-x})y'=y^2$

$$(e^x+e^{-x})\frac{dy}{dx}=y^2$$

$$\frac{dy}{dx}=\frac{y^2}{e^x+e^{-x}} \Rightarrow \frac{dx}{e^x+e^{-x}}=\frac{dy}{y^2} \quad \text{———} \quad (*)$$

$$\int\frac{dx}{e^x+e^{-x}}=\int\frac{dy}{y^2}$$

Now, $\dfrac{1}{e^x+e^{-x}}=\dfrac{e^x}{e^{2x}+1}=\dfrac{e^x}{1+(e^x)^2}$.

$$\therefore \int\frac{dx}{e^x+e^{-x}}=\int\frac{e^x}{1+(e^x)^2}dx=\tan^{-1}e^x+C$$

Substitute in (*) $\Rightarrow$ $\boxed{\tan^{-1}e^x+C=-\dfrac{1}{y}}$

8. verify that the following DE is Homogeneous, then solve the IVP:

$$xydx - (x^2 + y^2)dy = 0\,, \qquad y(1) = e.$$

$f(x,y) = xy$

$f(tx,ty) = (tx)(ty)$

$= t^2 xy$

$= t^2 f(x,y)$

$\therefore f$ is hom. of degree 2

$g(x,y) = -(x^2+y^2)$

$g(tx,ty) = -(x^2t^2+y^2t^2)$

$= t^2[-(x^2+y^2)]$

$= t^2 g(x,y)$

$\therefore g$ is hom of degree 2

$\therefore$ the given DE is homogeneous.

To solve: let $y = ux$, $dy = u\,dx + x\,du$

Substitute in the DE:

$x^2u\,dx - (x^2+x^2u^2)(u\,dx + x\,du) = 0$

$(\cancel{x^2u} - \cancel{x^2u} - x^2u^3)dx + (-x^3 - x^3u^2)du = 0$

$-x^2u^3\,dx - x^3(1+u^2)\,du = 0$

$\div x^3u^3$ $\quad -\dfrac{dx}{x} - \dfrac{1+u^2}{u^3}du = 0$

$\displaystyle\int\frac{dx}{x} = -\int\frac{1+u^2}{u^3}du$

$\displaystyle\int\frac{dx}{x} = -\int\left(\frac{1}{u^3} + \frac{1}{u}\right)du$

$\ln|x| = \dfrac{1}{2u^2} - \ln|u| + C$

$\ln|x| + \ln|u| - \dfrac{1}{2u^2} = C$

$\ln|x| + \ln\left|\dfrac{y}{x}\right| - \dfrac{x^2}{2y^2} = C$

$\ln|y| - \dfrac{x^2}{2y^2} = C$

using $y(1)=e \Rightarrow 1 - \dfrac{1}{2e^2} = C$

$\therefore$ the solution is

$$\boxed{\ln|y| - \frac{x^2}{2y^2} = 1 - \frac{1}{2e^2}}$$

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9. Solve: $(e^x + y)dx + (x + \ln y)dy = 0$.

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ It is an exact equation

$\Rightarrow \exists f$ such that

$$\frac{\partial f}{\partial x} = M = e^x + y, \quad \frac{\partial f}{\partial y} = N = x + \ln y$$

$$f(x,y) = \int (e^x + y)\,dx = e^x + xy + g(y) \quad \text{---} (*)$$

$$\frac{\partial f}{\partial y} = x + g'(y) = N = x + \ln y$$

$$\Rightarrow g'(y) = \ln y$$

$$\Rightarrow g(y) = \int \ln y\,dy = y\ln y - y + K$$

Substitute for $g(y)$ in $(*)$:

$$f(x,y) = e^x + xy + y\ln y - y + K$$

$\therefore$ the Solution is:

$$\boxed{e^x + xy + y\ln y - y = C}$$