KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 202-(072)

Second Major Exam

April 29, 2008

Time: 75 Minutes

I.D.: Name: Solution Sec.# Ser.#

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points
1	/16
2	/20
3	/24
4	/20
5	/20
Total	/100

Instructor: M. Samman

1. **(a)** Find the Singular Points of the ODE:

$$x^2(x^2-25)y''-(x^3+5x^2)y'+5x(x-5)^2y=0$$

The singular points of this equation are solutions of

$x^2(x^2-25)=0$

$\Rightarrow \quad x = 0, 5, -5$

(b) Find the radius of convergence and interval of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{3^n}{n!}x^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{n+1}}{(n+1)!}x^{n+2}}{\frac{3^n}{n!}x^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{3}{n+1}\right|$$

$$= 0$$

$\therefore$ The series is absolutely convergent on $(-\infty,\infty)$

So $R=\infty$

2. Find the recurrence relation for the series solutions of the DE $y''+2xy'+2y=0$ about $x_0=0$.

$$y=\sum_{n=0}^{\infty} c_n x^n \quad , \quad y'=\sum_{n=1}^{\infty} n c_n x^{n-1} \quad , \quad y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting in the DE,

$$y''+2xy'+2y=$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\text{Put } k=n-2} + \underbrace{\sum_{n=1}^{\infty} 2n c_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{k=n} = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} 2c_k x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k + \qquad + 2c_0 + \sum_{k=1}^{\infty} 2c_k x^k = 0$$

$$2c_2+2c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + 2kc_k + 2c_k\right] x^k = 0$$

$$\therefore 2c_2+2c_0=0 \quad \& \quad (k+2)(k+1)c_{k+2} + 2(k+1)c_k = 0$$

$$\Rightarrow c_2=-c_0 \quad \& \quad c_{k+2} = -\frac{2}{k+2}c_k \quad , \quad k=1,2,3,\ldots$$

3. (a) Show that $1, \cos 2x, \sin 2x$ form a fundamental set of solutions for the DE $y''' + 4y' = 0$. Also write the general solution of this equation.

First consider $y = 1$, then easily $y''' + 4y' = 0$. So $y=1$ is a solution

Next, consider $y = \cos 2x$. Then

$$y' = -2 \sin 2x \quad , \quad y'' = -4 \cos 2x \quad , \quad y''' = 8 \sin 2x$$

Substituting in the DE, we get

$$y''' + 4y' = 8 \sin 2x + 4(-2 \sin 2x) = 0$$

So $y = \cos 2x$ is a solution for the given DE.

Similarly one can check that $y = \sin 2x$ is a solution.

Now, we compute the Wronskain of these solutions:

$$W(1, \cos 2x, \sin 2x) = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2 \sin 2x & 2 \cos 2x \\ 0 & -4 \cos 2x & -4 \sin 2x \end{vmatrix}$$

$$= 8$$

$$\neq 0$$

Thus, $1, \cos 2x, \sin 2x$ are linearly independent, and so they form a fundamental set of solutions.

The general solution is $y = c_1 + c_2 \cos 2x + c_3 \sin 2x$.

(b) If $y_{p_1} = -\frac{1}{4}e^{-2x}$ is a particular solution of the DE $y'' + 2y' - 8y = 2e^{-2x}$, and $y_{p_2} = \frac{1}{9}e^{-x}$ is a particular solution of $y'' + 2y' - 8y = -e^{-x}$, obtain a solution for the DE $y'' + 2y' - 8y = -8e^{-2x} - 2e^{-x}$. **(Do not solve the equation)**

$y_{p_1} = -\frac{1}{4}e^{-2x}$ is a particular solution of $y'' + 2y' - 8y = 2e^{-2x}$ — (1)

$y_{p_2} = \frac{1}{9}e^{-x}$ " " " " $y'' + 2y' - 8y = -e^{-x}$ — (2)

To obtain a solution for the DE $y'' + 2y' - 8y = -8e^{-2x} - 2e^{-x}$, — (*)

we observe that the L.H.S. of this equation is the same as the L.H.S. of the above two equations, and the R.H.S. of (*) is the sum of -4(R.H.S. of (1)) $+ 2$(R.H.S. of (2)), which is

$$-4(2e^{-2x}) + 2(-e^{-x}) = -8e^{-2x} - 2e^{-x}$$

Hence, by the superposition principle, we deduce that

$y = -4(-\frac{1}{4})e^{-2x} + 2(\frac{1}{9})e^{-x}$ is a solution of (*)

i.e. $y = e^{-2x} + \frac{2}{9}e^{-x}$

(c) Solve the DE $y''' - 3y'' + 4y = 0$

The characteristic equation is $\lambda^3 - 3\lambda^2 + 4 = 0$

$\Rightarrow (\lambda+1)(\lambda^2 - 4\lambda + 4) = 0$

$(\lambda+1)(\lambda-2)^2 = 0$

$\lambda = -1, 2, 2$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\therefore$ the solution is:

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$$

4. Use the method of undetermined coefficients to solve the DE

$$y'' + 4y = 4\cos x + 3\sin x - 8$$

First we solve the associated homogeneous DE:

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i$$

$y_H = C_1 \cos 2x + C_2 \sin 2x$ - - - - - - - - - (1)

Write the given DE as $(D^2+4)y = 4\cos x + 3\sin x - 8$

$\text{Ann}(R.H.S.) = D(D^2+1)$

$D(D^2+1)(D^2+4)y = D(D^2+1)(4\cos x + 3\sin x - 8)$

$D(D^2+1)(D^2+4)y = 0$

Solving this equation $\Rightarrow$ $\lambda(\lambda^2+1)(\lambda^2+4) = 0 \Rightarrow \lambda = 0, \pm i, \pm 2i$

The solution is $y = C_1 + C_2\cos x + C_3 \sin x + \underbrace{C_4 \cos 2x + C_5 \sin 2x}_{y_H}$ ---(2)

Comparing (1) & (2), we get

$y_p = A + B\cos x + C\sin x \Rightarrow y_p' = -B\sin x + C\cos x \Rightarrow y_p'' = -B\cos x - C\sin x$

Substitute in the given DE: $y_p'' + 4y_p = 4\cos x + 3\sin x - 8$

$-B\cos x - C\sin x + 4A + 4B\cos x + 4C\sin x = 4\cos x + 3\sin x - 8$

$3B\cos x + 3C\sin x + 4A = 4\cos x + 3\sin x - 8$

Equating coeff's $\Rightarrow$ $4A = -8 \Rightarrow \boxed{A = -2}$, $3B = 4 \Rightarrow \boxed{B = \frac{4}{3}}$, $3C = 3 \Rightarrow \boxed{C = 1}$

$\therefore y_p = -2 + \frac{4}{3}\cos x + \sin x$

The solution is $y = y_H + y_p$

$= C_1 \cos 2x + C_2 \sin 2x + \frac{4}{3}\cos x + \sin x - 2$

5. Use variation of parameters to solve $y'' + y = \sec x$

First we solve the associated hom. equation $y'' + y = 0$.

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \qquad [\alpha = 0, \beta = 1]$$

$$y_H = C_1 \cos x + C_2 \sin x \qquad \text{so } \begin{array}{l} y_1 = \cos x \\ y_2 = \sin x \end{array}$$

We are seeking a particular solution for the given DE; $y_p = u_1 \cos x + u_2 \sin x$.

$$W = W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$u_1' = \frac{W_1}{W} = -\tan x$$

$$u_2' = \frac{W_2}{W} = 1$$

$$u_1 = \int -\tan x \, dx = \ln|\cos x|$$

$$u_2 = x$$

$$\therefore y_p = u_1 \cos x + u_2 \sin x$$

$$= \cos x \ln|\cos x| + x \sin x$$

$$y = y_H + y_p = C_1 \cos x + C_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$