Series Solution about regular singular points

<u>Part (1)</u> Difference of the Roots of Indicial Equation is Not an Integer

Use the Method of Frobenius to solve

$$2x^2y'' + xy' - (1+x)y = 0$$

<u>Step 1</u>: Write the Equation in the Form:

$$y'' + P(x)y' + Q(x)y = 0$$

Hence, $x = 0$ is a regular singular point for the given DE.

$\Rightarrow$ we can find at least one series solution of the form

$$y(x;r) = \sum_{k=0}^{\infty} c_n x^{n+r}$$. Therefore,

<u>Step 2</u>

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \Rightarrow -(1+x)y = -\sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1}$$

$$n+r$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \qquad \Rightarrow xy' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \Rightarrow 2x^2y'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r}$$

<u>Step 3</u>: Change of Power is Required only in one Series, i.e. (*): [k= n+1]

<u>Step 4</u>: Substitution into the DE gives:

$$\sum_{k=0}^{\infty} 2(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)c_k x^{k+r} - \sum_{k=0}^{\infty} c_k x^{k+r} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r} =$$

<u>Step 5</u>: Separate the terms for k = 0 in the 1st 3 Summations. Then Combine All:

$$[2r(r-1) + r - 1]c_0 + \sum_{k=1}^{\infty} \{[2(k+r)(k+r-1) + (k+r) - 1]c_k - c_{k-1}\}x^{k+r} = 0$$

<u>Note</u>: $2r(r-1) + r - 1 = 0$ is the Indicial Equation & $c_0 \neq 0$

Step 6: Recurrence Relation:

$$(k+r-1)(2k+2r+1)c_k - c_{k-1} = 0 \qquad (**)$$

Case 1: $r = 1$ in (**)

$$c_k = \frac{1}{k(2k+3)} c_{k-1}, \quad k = 1, 2, 3\ldots$$

$k=1:\ c_1 = (1/1.5)c_0$

$k=2:\ c_2 = (1/2.7)c_1 = (1/[2!5.7])c_0$

$k=3:\ c_3 = (1/3.9)c_2 = (1/[3!5.7.9])c_0$

$k=4:\ c_4 = (1/4.11)c_3 = (1/[4!5.7.9.11])c_0$

$\vdots$

Therefore, the Frobenius Series Solution is:

$$y(x) = x\{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\}$$

$$= c_0 x[1 + \frac{x}{5} + \frac{x^2}{2!5.7} + \frac{x^3}{3!5.7.9} + \frac{x^4}{4!5.7.9.11} + \cdots]$$

First Solution is given by:

$$y_1(x) = x + \frac{x^2}{5} + \frac{x^3}{2!5.7} + \frac{x^4}{3!5.7.9} + \frac{x^5}{4!5.7.9.11} + \cdots$$

Case 2: $r = -½$ in (**)

$$c_k = \frac{1}{k(2k-3)} c_{k-1}, \quad k = 1, 2, 3\ldots$$

$k=1:\ c_1 = -c_0$

$k=2:\ c_2 = (1/2.1)c_1 = (-1/[2!])c_0$

$k=3:\ c_3 = (1/3.3)c_2 = (-1/[3!1.3])c_0$

$k=4:\ c_4 = (1/4.5)c_3 = (-1/[4!1.3.5])c_0$

$\vdots$

Therefore, the Frobenius Series Solution is:

$$y(x) = x^{-1/2}\{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\}$$

$$= c_0 x^{-1/2}[1 - x - \frac{x^2}{2!} + \frac{x^3}{3!1.3} - \frac{x^4}{4!1.3.5} + \cdots]$$

Second LI Solution is given by:

$$y_2(x) = x^{-1/2}[1 - x - \frac{x^2}{2!} + \frac{x^3}{3!1.3} - \frac{x^4}{4!1.3.5} + \cdots]$$

General Solution:

$$y(x) = a_1 y_1(x) + a_2 y_2(x)$$

Part (2) **Difference of the Roots of Indicial Equation is a Positive Integer**
Case (i) Two Frobenius Solutions

Use the Method of Frobenius to get 2 LI Solutions of

$$xy'' + (x-6)y' - 3y = 0$$

about the R S P $x_0 = 0$.

Step 1: Write the Equation in the Form:

$$x^2 y'' + xA(x)y' + B(x)y = 0$$

Here, $A(x) = x - 6;\quad B(x) = -3x$ both are Analytic at $x_0 = 0$.

Step 2: Write the Indicial Equation:

$$r(r-1) + A(0)r + B(0) = 0$$

$$r(r-1) + (-6)r + (0) = 0 \Rightarrow r^2 - 7r = 0 \Rightarrow r(r-7) = 0$$

Step 3: Roots of Indicial Equation: i) $r = 0$, ii) $r = 7$
[Real & Distinct Roots but Differ by an Integer.]
$\Rightarrow$ The Equation has at least one Frobenius Solutions

Step 4: Solution is of the form $y(x;r) = \sum_{n=0}^{\infty} c_n x^{n+r}$. Therefore,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \Rightarrow -3y = -\sum_{n=0}^{\infty} 3c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \Rightarrow (x-6)y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} 6(n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \Rightarrow xy'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1}$$

Step 5: Change of Power is Required in two Series, i.e. (*): [k = n-1]

Step 6: Substitution into the DE gives:

$$\sum_{k=-1}^{\infty}[(k+r+1)(k+r)-6(k+r+1)]c_{k+1}x^{k+r}+\sum_{k=0}^{\infty}[(k+r)-3]c_k x^{k+r}=0$$

Step 7: Separate the terms for k = -1 in the 1st Summation. The Combine All:

$$r(r-7)c_0x^{-1+r}+\sum_{k=0}^{\infty}\{[(k+r+1)(k+r-6)]c_{k+1}+(k+r+1)c_k\}x^{k+r}=0$$

[Note: $r(r-7)=0$ is the Indicial Equation & $c_0 \neq 0$] $r=0,7$

Step 8: Recurrence Relation: k = 0,1,2,……

$$[(k+r+1)(k+r-6)]c_{k+1}+(k+r+1)c_k=0 \qquad (**)$$

Case 1:

For the Smaller Root $r = 0$ in (**)

$$(k+1)(k-6)c_{k+1}+(k-3)c_k=0, \quad k=0,1,2,3...$$

Important Note: k – 6 = 0 when k = 6, we can not divide by the term to write c_{k+1} in terms of c_k until k > 6.

$k=0:\ -6c_1+(-3)c_0=0 \Rightarrow c_1=-(1/2)c_0$

$k=1:\ -5c_2+(-2)c_1=0 \Rightarrow c_2=(1/10)c_0$

$k=2:\ -4c_3+(-1)c_2=0 \Rightarrow c_3=-(1/120)c_0$

$k=3:\ -3c_4+(0)c_3=0 \Rightarrow c_4=0$

$k=4:\ -2c_5+(1)c_4=0 \Rightarrow c_5=0$

$k=5:\ -1c_6+(2)c_5=0 \Rightarrow c_6=0$

$k=6:\ (0)c_7+(3)c_6=0 \Rightarrow (0)c_7=0$

$\Rightarrow c_7$ can be assigned any value.

For $k \geq 7$, we have the recurrence relation:

$$c_{k+1}=\frac{-(k-3)}{(k+1)(k-6)}c_k$$

$k=7:\ c_8=(-4/8.1)c_7$

$k=8:\ c_9=(-5/9.2)c_8=(4.5/[2!8.9])c_7$

$k=9:\ c_{10}=(-6/10.3)c_9=(-4.5.6/[3!8.9.10])c_7$

$\vdots$

(1) Choose $c_0 \neq 0$ and $c_7=0$. Then 1st. Sol. is:

$$y_1(x)=c_0\left[1-\frac{x}{2}+\frac{x^2}{10}-\frac{x^3}{120}\right]$$

(2) Choose $c_7 \neq 0$ and $c_0=0$. Then 2nd. Sol. is:

$$y_2(x)=\{c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots\}$$

Note:

1. In the problems of above type, first use the smaller root of the Indicial Eq. Sometimes it provides both LI Frobenius solutions.
2. If 2nd solution is not found as indicated in the above note, the apply the Method of Reduction of Order using $y_2=u(x)y_1$ and find $u(x)$ as indicated in the Method above.

<u>Part (3)</u> Identical Roots of the Indicial Equation
(1^{st} Frobenius Solution, 2^{nd} by Reduction of Order)

Use the Method of Frobenius to get 2 LI Solutions of

$$4x^2y'' - 4x^2y' + (1+2x)y = 0$$

about the R S P $x_0 = 0$.

<u>Step 1</u>: Write the Equation in the Form:

$$x^2y'' + xA(x)y' + B(x)y = 0$$

Here, $A(x) = -x$; $B(x) = (1+2x)/4$ both are Analytic at $x_0 = 0$.

<u>Step 2</u>: Write the Indicial Equation:

$$r(r-1) + A(0)r + B(0) = 0$$

$$r(r-1) + (0)r + (1/4) = 0 \Rightarrow r^2 - r + (1/4) = 0 \Rightarrow (r-(1/2))^2 = 0$$

<u>Step 3</u>: Roots of Indicial Equation: i) $r = 1/2$, ii) $r = 1/2$
[Real & Identical Roots.]

$\Rightarrow$ The Equation has at least one <u>Frobenius Solution</u>

<u>Step 4</u>: Solution is of the form $y(x;r) = \sum_{n=0}^{\infty} c_n x^{n+r}$. Therefore,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \Rightarrow (1+2x)y = \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} 2c_n x^{n+r+1} \, *$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \qquad \Rightarrow -4x^2y' = -\sum_{n=0}^{\infty} 4(n+r)c_n x^{n+r+1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \Rightarrow 4x^2y'' = \sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r}$$

<u>Step 5</u> Change of Power is Required in two Series, i.e. (*): [k= n+1]

<u>Step 6</u>: Substitution into the DE gives:

$$\sum_{k=0}^{\infty}[4(k+r)(k+r-1)+1]c_k x^{k+r} - \sum_{k=1}^{\infty}[4(k+r-1)-2]c_{k-1}x^{k+r} = 0$$

Step 7: Separate the terms for k = 0 in the 1st Summation. Then Combine All:

$$[4r(r-1)+1]c_0 x^r + \sum_{k=1}^{\infty}\{[4(k+r)(k+r-1)+1]c_k - [4(k+r-1)-2]c_{k-1}\}x^{k+r} =$$

[Note: $4r^2 - 4r + 1 = 0$ is the Indicial Equation & $c_0 \neq 0$]

Step 8: Recurrence Relation: $n = 0,1,2,\ldots\ldots$

$$[4(k+r)(k+r-1)+1]c_k - [4(k+r-1)-2]c_{k-1} = 0$$

(**)

Case 1: $r = 1/2$ in (**)

$$c_k = \frac{k-1}{k^2}c_{k-1}, \quad k = 1,2,3\ldots$$

$k = 1$: $c_1 = 0 \Rightarrow c_k = 0,\ k = 1,2,\cdots$

Therefore, the Frobenius Series Solution is:

$$y(x) = x^{1/2}\{c_0 x + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\}$$
$$= c_0 x^{1/2}$$

First Solution is given by: $y_1(x) = x^{1/2}$

Important Note: We can not find the Second Frobenius Solution for the DE. For Second Solution we shall have to use the Method of Reduction of Order.

Second Solution will be of the Form

$$y_2 = y_1 \int \frac{G(x)}{[y_1]^2}dx \quad \text{where } G(x) = e^{-\int [P(x)/x]dx}$$

Here: $\mathrm{P}(x) = -x$ and $[y_1]^2 = x$

$$\Rightarrow \int \frac{e^{-\int [P(x)/x]dx}}{[y_1]^2}dx = \int \frac{e^x}{x}dx = \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^n}{n!}dx$$

$$= \int \sum_{n=0}^{\infty}\frac{x^{n-1}}{n!}dx = \int \frac{1}{x}dx + \int \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}dx$$

$$= \ln x + \sum_{n=1}^{\infty}\frac{x^n}{n!n}$$

$$\Rightarrow y_2 = y_1\left[\ln x + \sum_{n=1}^{\infty}\frac{x^n}{n!n}\right]$$

General Solution:

$$y(x) = a_1 y_1(x) + a_2 y_2(x)$$