KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 260-(071)

Second Major Exam

December 5, 2007

Time: 100 Minutes

Name: Solution Ser.#

Show All Necessary Work

Question	Points
1	/16
2	/6
3	/11
4	/6
5	/8
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8	/7
Total	/70

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1. Solve the following DEs

8 **(a)** $y''' - 3y'' + 4y = 0$

The auxiliary equation is $\lambda^3 - 3\lambda^2 + 4 = 0$

$$\Rightarrow (\lambda+1)(\lambda^2 - 4\lambda + 4) = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$$(\lambda+1)(\lambda-2)^2 = 0$$

$$\lambda = -1, 2, 2.$$

$\therefore$ the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$$

8 **(b)** $y'' + 4y' + 9y = 0$

The auxiliary equation is $\lambda^2 + 4\lambda + 9 = 0$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-36}}{2} = -2 \pm \sqrt{5}\, i$$

$$\downarrow$$

$$\alpha = -2, \beta = \sqrt{5}$$

Hence the solution is

$$y = e^{-2x}\left[c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x\right].$$

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2. Define each one of the following:

2 (a) Vector space

See your notes

2 (b) Subspace

2 (c) Dimension of a vector space.

3\. (a) Let $B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 8 & 4 \end{bmatrix}$ Find a matrix D so that $B = DB$

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Note that $\underset{2\times 3}{B} = \underset{\boxed{2\times 2}}{D}\,\underset{2\times 3}{B}$. So let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B = DB \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 8 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 8 & 4 \end{bmatrix} = \begin{bmatrix} a & -2a+8b & a+4b \\ c & -2c+8d & c+4d \end{bmatrix}$

$\Rightarrow a = 1,\ c = 0,\ b = 0,\ d = 1$

$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & -2 \\ 1 & 4 & -2 & 4 \end{bmatrix}$

(b) Find $2A^T + I_4$.

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$A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 1 & -1 & 1 & -2 \\ 0 & 1 & -2 & 4 \end{bmatrix}$

$2A^T + I_4 = \begin{bmatrix} 2 & 0 & 2 & 2 \\ 4 & 2 & 6 & 8 \\ 2 & -2 & 2 & -4 \\ 0 & 2 & -4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 4 & 3 & 6 & 8 \\ 2 & -2 & 3 & -4 \\ 0 & 2 & -4 & 9 \end{bmatrix}$

(c) Find $det(A)$.

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$\det A = \left| \quad \right| = \begin{vmatrix} 0 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & -2 \\ 1 & 4 & -2 & 4 \end{vmatrix} = 1\begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & -2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - 2\left(-1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}\right) = 6 - 3 - 2\left[-1(-3) - 1(1)\right]$

$= -1$

4. (a) Are the vectors $u = (1, 2, 3)$, $v = (3, 2, 1)$, $w = (1, -1, 6)$, linearly independent?

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Note that $u, v, w \in \mathbb{R}^3$. We compute the following determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 3 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 13 - 3(15) + 1(-4)$$

$$= -36$$

$$\neq 0$$

Hence, the vectors u, v, w are linearly independent

(b) Write the vector w as a linear combination of the standard basis in R^3.

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$$w = (1, -1, 6) = (1,0,0) + (0,-1,0) + (0,0,6)$$

$$= (1,0,0) - (0,1,0) + 6(0,0,1)$$

$$\therefore w = e_1 - e_2 + 6e_3$$

5. (a) Let $W = \{(x_1, x_2, x_3, x_4) : x_4 \geq 0\}$. Prove or disprove that W is a subspace of the vector space R^4.

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Let $u, v \in W$, so

$$u = (x_1, x_2, x_3, x_4), \text{ with } x_4 \geqslant 0$$
$$v = (y_1, y_2, y_3, y_4), \text{ with } y_4 \geqslant 0 \quad \text{Then}$$

$u+v = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4) \in W$. Since $x_4+y_4 \geqslant 0$.

Now, if c is a negative scalar, say $c = -1$, then $c.u \notin W$

for example take $u = (1,2,3,4) \in V$, $c = -1$, then

$$cu = -1(1,2,3,4) = (-1,-2,-3,-4) \notin W \text{ since } -4 \ngeq 0$$

Hence W is not a subspace of R^4.

(b) Let $v_1 = (1,2,1)$, $v_2 = (1,0,2)$, $v_3 = (1,1,0)$. Does the set $\{v_1, v_2, v_3\}$ form a basis for R^3?

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Since v_1, v_2, v_3 are 3 vectors in R^3, all what we need is to check whether they are linearly independent or not. For this, we compute

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 3 \neq 0$$

Hence v_1, v_2, v_3 form a basis for R^3.

6. (a) If $S = \{u_1, u_2, \dots, u_k\}$ is a linearly independent set of vectors. Show that any vector $w \in Span\, S$ can be expressed in a unique way as a linear combination of the vectors $u_1, u_2, \dots, u_k$.

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Let $w \in Span\, S$. Then

$$w = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k$$

Suppose also, $w = b_1 u_1 + b_2 u_2 + \cdots + b_k u_k$. Then

$$0 = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_k - b_k)u_k$$

But $u_1, u_2, \dots, u_k$ are linearly independent, thus

$a_1 - b_1 = 0$, $a_2 - b_2 = 0$, $\dots$, $a_k - b_k = 0$,

i.e. $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_k = b_k$,

which means that w has a unique representation as a linear comb. of $u_1, u_2, \dots, u_k$

(b) Let A be $n \times n$ matrix. Show that the set of all vectors X such that $AX = 7X$ is a subspace of R^n.

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Let W be the set of all vectors X such that $AX = 7X$.

Let $X_1, X_2 \in W$. Then $AX_1 = 7X_1$ and $AX_2 = 7X_2$.

Consider $A(X_1 + X_2) = AX_1 + AX_2 = 7X_1 + 7X_2 = 7(X_1 + X_2)$

$\Rightarrow X_1 + X_2 \in W$.

Next, if c is a scalar and $X \in W$, then

$$A(cX) = c(AX) = c(7X) = (c7)X = (7c)X = 7(cX)$$

Hence $cX \in W$.

This shows that W is a subspace of R^n.

7. Find a basis and the dimension of the solution space of the system.

$$\begin{aligned} x_1 + 5x_2 + 13x_3 + 14x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \\ 2x_1 + 7x_2 + 17x_3 + 19x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 2 & 5 & 11 & 12 & 0 \\ 2 & 7 & 17 & 19 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 5 & 13 & 14 \\ 0 & -5 & -15 & -16 \\ 0 & -3 & -9 & -9 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[-5R_3+R_1]{5R_3+R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\Rightarrow x_4 = 0, \quad x_3 = t, \quad x_2 = -3t, \quad x_1 = 2t$$

$\therefore$ the solution to the system is $\begin{bmatrix} 2t \\ -3t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Hence $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the solution space of the system,

and clearly this solution space is one-dimensional.

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8. Write True or False for each of the following:

(a) An $n \times n$ matrix B is invertible if and only if $det\, B \neq 0$. (✓)

(b) The system $AX = B$ has a unique solution if and only if A is non-singular.

(✓)

(c) If v_1, v_2, v_3, v_4, v_5 are linearly independent, then they are distinct. (✓)

(d) Two basis for a vector space must have the same number of vectors. (✓)

(e) If v_1, v_2, v_3, v_4, v_5 are vectors in R^4, then they are linearly independent.

(X)

(g) For any continuous function f, the initial value problem:

$$(x-1)y'' + 3y' - 6y = f(x), \qquad y(1) = 3,\ y'(1) = 5$$

must have a unique solution on R. (X)

(h) Any set contains the zero vector must be linearly dependent. (✓)