KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 260-(071)

First Major Exam

October 31, 2007

Time: 90 Minutes

Name: Solution Ser.#

Show All Necessary Work

Question	Points
1	/10
2	/11
3	/11
4	/11
5	/11
6	/10
7	/6
Total	/70

1. (a) For each of the following, state whether the equation is linear or nonlinear, and give its order.

Equation	Linearity	Order
$yy' + 6y + 3 - x^2 = 0$	Non-linear	1
$(\cos x)y'''' - (\sin x)y' - 8 = 0$	Linear	3
$y' = 1 - xy + y^2$	Non-linear	1
$x(y'')^3 + (y')^4 - y = 0$	Non-linear	2

$y\,y'$

y^2

$(y'')^3$

(b) Check the following 1st order ODEs and write in the brackets *S* for *Separable*, *L* for *Linear*, *E* for *Exact*, *H* for *Homogeneous*, and *B* for *Bernoulli*:

i. $(y - xy^2)dy = ydx$ (L) in x

ii. $(e^{y/x} + e^{x^3/y^3} + 1)dy = (1 + \ln(y/x))dx$ (H)

iii. $(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$ (E)

iv. $y^3 \frac{dy}{dx} = (y^4 + 1)\cos x$ (S)

v. $(x^2 y - y^3)dx = x^3 dy$ (H), B

vi. $(x^2 - 4x - 3xy + 4)dx = (x^2 - x - 2)dy$ (L)

2. Show that the following differential equation is exact and solve it:

$$y' = \frac{e^x \cos y - 2xy}{e^x \sin y + x^2}$$

$$\frac{dy}{dx} = \frac{e^x \cos y - 2xy}{e^x \sin y + x^2}$$

$$\underbrace{\left(e^x \cos y - 2xy\right)}_{M} dx + \underbrace{\left(-e^x \sin y - x^2\right)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = -e^x \sin y - 2x \,, \quad \frac{\partial N}{\partial x} = -e^x \sin y - 2x$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Longrightarrow \text{Exact}.$$

$$\therefore \frac{\partial f}{\partial x} = M = e^x \cos y - 2xy \,, \quad \frac{\partial f}{\partial y} = N = -e^x \sin y - x^2$$

$$f(x,y) = \int \left(e^x \cos y - 2xy\right) dx = e^x \cos y - x^2 y + g(y)$$

$$\frac{\partial f}{\partial y} = -e^x \sin y - x^2 + g'(y) = N$$

$$\Rightarrow -e^x \sin y - x^2 + g'(y) = -e^x \sin y - x^2$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = C_1$$

$$\therefore f(x,y) = e^x \cos y - x^2 y + C_1$$

the solution is $e^x \cos y - x^2 y = C$

3. Solve the following initial value problem:

$y' = (y \sec x)^2, \quad y(\frac{\pi}{4}) = 3$

$$y' = y^2 \sec^2 x$$

$$\frac{dy}{dx} = y^2 \sec^2 x$$

$$\frac{dy}{y^2} = \sec^2 x \, dx$$

$$\int \frac{dy}{y^2} = \int \sec^2 x \, dx$$

$$-\frac{1}{y} = \tan x + C$$

$$y = \frac{-1}{\tan x + C}$$

Use the initial condition $y(\frac{\pi}{4}) = 3$:

$$\Rightarrow 3 = \frac{-1}{1+C} \qquad \Rightarrow C = \frac{-4}{3}$$

Hence, the solution is

$$y = \frac{-1}{\tan x - \frac{4}{3}}$$

4. Write the following DE as **Bernoulli** equation and then solve the IVP:

$$xy' - y - \frac{1}{y^2} = 0, \quad y(1) = 0$$

$$y' - \frac{1}{x} y = \frac{1}{xy^2}$$

$$y' - \frac{1}{x} y = \frac{1}{x} y^{-2} \quad \longleftarrow \text{Bernoulli}$$

put $w = y^{1+2} = y^3 \Rightarrow y = w^{\frac{1}{3}}, \quad \frac{dy}{dx} = \frac{1}{3} w^{-\frac{2}{3}} \frac{dw}{dx}, \quad y^{-2} = w^{-\frac{2}{3}}$

Substituting we get:

$$\frac{1}{3} w^{-\frac{2}{3}} \frac{dw}{dx} - \frac{1}{x} w^{\frac{1}{3}} = \frac{1}{x} w^{-\frac{2}{3}}$$

$$\frac{dw}{dx} - \frac{3}{x} w = \frac{3}{x} \quad \longleftarrow \text{Linear}$$

$$\text{I.F.} = e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \frac{1}{x^3}$$

$$\frac{d}{dx}\left(\frac{w}{x^3}\right) = \frac{3}{x^4}$$

$$\frac{w}{x^3} = \int \frac{3}{x^4} dx = -x^{-3} + c$$

$$w = -1 + cx^3$$

$y^3 = -1 + cx^3$. Now $y(1) = 0 \Rightarrow 0 = -1 + c \Rightarrow \boxed{c = 1}$

$$y^3 = -1 + x^3$$

$$\therefore y = \sqrt[3]{x^3 - 1}$$

5. Solve the following system:

$$\begin{aligned} 5x_1 + 7x_2 + 4x_3 + x_4 &= 5 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 &= 3 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \\ 1 & 1 & -2 & 3 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \\ 2 & 3 & 3 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[-2R_1 + R_3]{-5R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 2 & 14 & -14 & -15 \\ 0 & 1 & 7 & -7 & -5 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2} R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -15/2 \\ 0 & 1 & 7 & -7 & -5 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -15/2 \\ 0 & 0 & 0 & 0 & 5/2 \end{array}\right]$$

$\Rightarrow$ the system has no solution.

i.e. the system is inconsistent.

6. Find the inverse B^{-1}, if it exists, for the matrix $B = \begin{bmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & 7 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 7 & 3 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{-2R_1 + R_2}{-3R_1 + R_3} \qquad \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -3 & 1 \end{array}\right]$$

$$\frac{-3R_2 + R_1}{2R_2 + R_3} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -3 & 7 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{array}\right]$$

$$\frac{-5R_3 + R_1}{R_1 + R_2} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 42 & -5 \\ 0 & 1 & 0 & 3 & -9 & 1 \\ 0 & 0 & 1 & 2 & -7 & 1 \end{array}\right]$$

$$\equiv \qquad \left[\, \mathrm{I} \mid A^{-1} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

7. Mark (✔) or (×) for each of the following:

(a) The differential equation $(2y^2 + x^2)dx = xydy$ is homogeneous (✓)

(b) For any two n×n matrices A and B: $AB = AC \Rightarrow B = C$ (X)

(c) For any two n×n matrices A and B: $(A - B)(A + B) = A^2 - B^2$ (X)

(d) The homogeneous system AX = 0 is always consistent (✓)

(e) The function $y_1(x) = x^4 \ln x$ is a solution of the DE $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$ (✓)

(f) The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix (✓)