Math 202 Quiz # 9

Name: Solution Section # ____________ Sr. # ____________

Find the general solution of the system $X' = AX$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2-2\lambda+1) = (\lambda-1)^3 = 0$$

$\Rightarrow \lambda = 1, 1, 1$ eigen values.

$\underline{\lambda = 1}$

$$(A-\lambda I)K = 0$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} k_2 = k_3 \\ k_1 = 0 \end{array}\right\} \Rightarrow K = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$X_1 = Ke^{\lambda t} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t$$

To find second solution, we solve $(A-\lambda I)P = K$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} P_1 = 0 \\ P_2 = 1 + P_3 \end{array}\right\} \quad P_3 = 0 \Rightarrow \quad P = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = Kte^{\lambda t} + Pe^{\lambda t}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t}$$

Next, to find third solution, we solve $(A-\lambda I)Q = P$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} q_1 = \frac{1}{2} \\ q_2 = q_3 \end{array}\right\} \quad \text{Take } q_2 = 0 \text{, then} \quad Q = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \frac{t^2}{2} K e^{\lambda t} + t P e^{\lambda t} + Q e^{\lambda t}$$

$$= \frac{t^2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} e^{t}$$

The general solution of the given system is

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t e^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} \right) + c_3 \left(\frac{t^2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} e^{t} \right)$$