

Review of Power Series (II)

Function f is Analytic at a Point x_0 if

f can be represented by a power series in $x-x_0$ with +ive Radius of Convergence

$$\text{i.e. } f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

with Radius of convergence $R > 0$

Examples

- I. The functions e^x , $\cos x$, $\sin x$ and $1/(x-1)$ are analytic at $x_0 = 0$.
- II. $1/(x-1)$ is not analytic at $x_0 = 1$

Note

1. If $f(x)$ and $g(x)$ are analytic at x_0 , then so are $f(x) \pm g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ [Provided $g(x_0) \neq 0$]

2. A Rational Function $P(x)/Q(x)$ is Analytic at all Points $x = x_0$ provided that $Q(x_0) \neq 0$

3. Important Result

- Given:**
- i. Polynomials $P(x)$ and $Q(x)$
 - ii. $Q(x_0) \neq 0$
 - iii. R = Radius of the Power Series representing $P(x)/Q(x)$ about x_0

Conclusion: R will be the distance from x_0 to the nearest zero of $Q(x)$ in the Complex Plane.

Example: $P(x)/Q(x) = 1/(x^2 + 1)$ is analytic at $x_0 = 0$ (since $Q(0) \neq 0$). The nearest zeros of $Q(x)$ from $x_0 = 0$ in the Complex Plane are $\pm i$. Therefore, R , the radius of Convergence of the Power Series of $1/(x^2 + 1)$ is 1.

Exercise: Only find the Radius of Convergence of the Power Series Expansion of $f(x) = (1-x)/(x^2+2x+2)(x-2)$ centered (in each case) at $x_0 = 0, 3, 7$.

Ordinary and Singular Points of a 2nd Order ODE (in Standard Form)

$$y'' + P(x)y' + Q(x)y = 0 \dots (*)$$

1. The Point $x = x_0$ is called an **Ordinary Point** of the ODE (*) if both coefficient functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

Otherwise the point x_0 is called a **Singular Point** of the ODE (*).

Example

$$y'' + [1/x^2(x+5)]y' + [x/(x^2-1)]y = 0 \dots (1)$$

- i. All real numbers except $x = \pm 1, -5$ are the Ordinary Points of the ODE (1).
- ii. $x = \pm 1, -5$ are the Singular Points.

Regular and Irregular Singular Points of a 2nd Order ODE

(in Standard Form)

$$y'' + P(x)y' + Q(x)y = 0 \dots (*)$$

The Point $x = x_0$ is called a **Regular Singular Point** of the ODE (*) if

(a) x_0 is a Singular Point of the ODE (*)

- (b) Both functions
- $$A(x) = (x - x_0)P(x)$$
- $$B(x) = (x - x_0)^2Q(x)$$
- are analytic at $x = x_0$.

Otherwise the point x_0 is called an **Irregular Singular Point**

Example

$$y'' + [1/x^2(x+1)^2]y' + [x/(x^2-1)^2]y = 0 \dots (2)$$

1. $x = 1$ is a Regular Singular Points of ODE (2)
2. $x = 0, -1$ are the Irregular Singular Points of the ODE (2).

I. Series Solution about Ordinary Points (O P)

Existence of Power Series Solution

Given

- i. $x = x_0$ is an O P of the 2nd Order LDE:
 $y'' + P(x)y' + Q(x)y = 0 \dots (*)$
- ii. R is the distance from x_0 to the nearest singular point (Real or Complex) of the DE (*) in the Complex Plane.

Conclusion

- i. We can find Two LI Solutions of (*) in the form of a PS centered at x_0 :

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

- ii. The series will converge at least in the interval $|x - x_0| < R$.

Method of Solution when $x_0 = 0$ is Ordinary Point

- Find the Singular Points and the Radius of Convergence
- Write $y(x) = \sum_{n=0}^{\infty} c_n x^n$
- Find the Series for y' & y''
- Substitute in the ODE.
- Shift the Indices, if required, and compose the coefficients of same powers of x .
- Equate the Coeff. of the powers of x to 0.
- Find the Recurrence Relation.
- Find 2 Linearly Independent Solutions

Exercises

[Find Rad. of Conv. of Solution Series]

- Show that $(1+x^2)y'' + 3xy' + y = 0$ has two LI Series Solutions centered at Zero.
- Find 2 LI Series Solutions in Powers of x to
 - $y'' - 2xy' - 4y = 0$
 - $y'' + (1+x^2)y' - 8xy = 0$

II. Series Solution about Regular Singular Points (R S P)

Method of Frobenius

Given

- i. $x = x_0$ is a R S P of the 2nd Order LDE:
 $y'' + P(x)y' + Q(x)y = 0 \dots (#)$
- ii. R is the distance from x_0 to the nearest singular point (Real or Complex) of the DE (*) in the Complex Plane.

Conclusion

- i. There exists at least one Solution of (#) in the form of Series with some constant r :

$$y(x; r) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \dots (**)$$

- ii. The series will converge at least in the annulus $0 < |x - x_0| < R$.

A Note on the Constant r in (**) "Indicial Equation"

[For Simplicity we consider the RSP $x_0 = 0$]

- The constant r in (**) is called the Exponent of Singularity.
- Set $A(x) = xP(x)$; $B(x) = x^2h(x)$.
 $\Rightarrow A(x)$ & $B(x)$ will be analytic at $x_0 = 0$
 $\Rightarrow A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$
- We can write Eq. (#) as
 $x^2 y'' + xA(x)y' + B(x)y = 0 \dots (**)$
- Note:** Eq. (##) becomes Cauchy-Euler Eq. if $A(x)$ & $B(x)$ are constants. In this case, the Eq. (##) has a solution of the form $y = x^m$. Here, m is a root of " $m(m-1) + cm + d = 0$ ". It may be -ive or a fraction. In such case, the series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ will not match with $y = x^m$.

- Remedy:** To include these possibilities in the series solution, we introduce additional parameter r in the proposed series and try $y(x; r) = \sum_{n=0}^{\infty} c_n x^{n+r}$.
- This substitution in (##) after eliminating x^r , provides us the following Eq. corresponding to constant term:
 $r(r-1) + A_0 r + B_0 = 0$; $A_0 = A(x_0)$, $B_0 = B(x_0)$
 This Eq. is known as "**Indicial Eq.**". Its solutions help us overcome the difficulty pointed in (4).

Method for Series Solution About R S P $x_0 = 0$

Part I: Roots of Indicial Eq.

Step 1: Check that $x_0 = 0$ is an R S P.

Step 2: Find the Indicial Equation.

(a) Write the DE in the Form:

$$x^2 y'' + x A(x) y' + B(x) y = 0$$

(b) The Indicial Equation is:

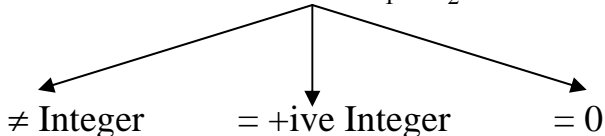
$$r(r-1) + A_0 r + B_0 = 0;$$

where $A_0 = A(x_0)$, $B_0 = B(x_0)$

Step 3: Find the Roots r_1 & r_2 of the Indicial Equation. ($r_1 \geq r_2$)

Part II : Nature of Roots r_1 & r_2

3 cases for $r_1 - r_2$



Series Solution

Part III : Case (i) $r_1 - r_2 \neq$ Integer

i. There will be two LI solutions of the Form:

$$y(x; r_1) = \sum_{n=0}^{\infty} c_n x^{n+r_1}; \quad y(x; r_2) = \sum_{n=0}^{\infty} d_n x^{n+r_2}$$

i. Find for each series the constants c_n and d_n separately following the method of Series Solution about Ordinary Points.

Part III : Case (ii) $r_1 - r_2 =$ +ive Integer

i. One Solution will be of the form

$$y(x; r_1) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

ii. 2nd Solution (Using Reduction of Order)

$$y_2(x) = u(x) y(x; r_1)$$

where

$$u(x) = \int \frac{e^{-\int [A(x)/x] dx}}{[y(x; r_1)]^2} dx$$

Note: In some cases, the solution corresponding to a smaller root provides two LI PS Solutions.

Important Steps for Finding of $u(x)$

Step 1: Evaluate $G(x) = e^{-\int [A(x)/x] dx}$

Step 2: Find the Power Series of $G(x)$:

If we know the Power Series of $A(x)$ then
(Using Term by Term Integration)

$$\int [A(x)/x] dx = A_0 \ln x + A^*(x)$$

where:
$$A^*(x) = \sum_{j=1}^{\infty} \frac{A_j}{j} x^j$$

[A_j = Coefficients in the Power Series of $A(x)$]

Step 3: Find a Few Terms the Power Series of $[y(x; r_1)]^2$

Step 4: Use Long Division to find a Few Terms the Power Series of

$$1 / [y(x; r_1)]^2$$

Step 5: Find the Power Series of the Product of the two Power Series obtained in Step 2 and Step 4. [Multiply the terms of the both PS]

Step 6: Use Term by Term Integration for the Power Series obtained in Step 5. This gives us the Power Series of $u(x)$.

Part III : Case (iii) $r_1 - r_2 = 0$

Here, the method for finding the 2 LI Solutions is exactly identical to that followed in Part III: Case (ii).

General Form of the 2nd LI Solution

Case (ii)

$$y_2 = C y(x; r_1) \ln x + \sum_{n=0}^{\infty} d_n x^{n+r_2}$$

[Here, the Constant C may be 0, but $d_0 \neq 0$]

Case (iii)

$$y_2 = y(x; r_1) \ln x + \sum_{n=1}^{\infty} d_n x^{n+r_2}$$

