Review of Power Series (II)

Function f *is* Analytic at a Point x_0 if

f can be represented by a power series in $x-x_0$ with +ive Radius of Convergence

i.e. $f(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^n$

with Radius of convergence $R > 0$

Examples

I. The functions e^x, $\cos x$, $\sin x$ and $1/(x-1)$ are analytic at $x_0 = 0$.

II. $1/(x-1)$ is not analytic at $x_0 = 1$

Note

1. If $f(x)$ and $g(x)$ are analytic at x_0, then so are $f(x) \pm g(x)$, $f(x)\,g(x)$, and $f(x)/g(x)$ [Provided $g(x_0) \neq 0$]

2. A Rational Function $P(x)/Q(x)$ is Analytic at all Points $x = x_0$ provided that $Q(x_0) \neq 0$

3. Important Result

Given: i. Polynomials $P(x)$ and $Q(x)$

ii. $Q(x_0) \neq 0$

iii. $\boldsymbol{R}$= Radius of the Power Series representing $P(x)/Q(x)$ about x_0

Conclusion: $\boldsymbol{R}$ will be the distance from x_0 to the nearest zero of $Q(x)$ in the Complex Plane.

Example: $P(x)/Q(x) = 1/(x^2 + 1)$ is analytic at $x_0 = 0$ (since $Q(0) \neq 0$). The nearest zeros of $Q(x)$ from $x_0 = 0$ in the Complex Plane are $\pm\, i$. Therefore, $\boldsymbol{R}$, the radius of Convergence of the Power Series of $1/(x^2 + 1)$ is 1.

Exercise: Only find the Radius of Convergence of the Power Series Expansion of $f(x) = (1-x)/(x^2+2x+2)(x-2)$ centered (in each case) at $x_0 = 0, 3, 7$.

Ordinary and Singular Points of a 2nd Order ODE (in Standard Form)

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \ldots. \quad (*)$$

1. The Point $\boldsymbol{x} = \boldsymbol{x_0}$ is called an ***Ordinary Point*** of the ODE (*) if both coefficient functions $P(x)$ and $Q(x)$ are analytic at $\boldsymbol{x} = \boldsymbol{x_0}$.

Otherwise the point $\boldsymbol{x_0}$ is called a **Singular Point** of the ODE (*).

Example

$$y'' + [1/x^2(x+5)]y' + [x/(x^2-1)]\,y = 0 \ldots (1)$$

i. All real numbers except $\boldsymbol{x} = \pm1, -5$ are the Ordinary Points of the ODE (1).

ii. $\boldsymbol{x} = \pm1, -5$ are the Singular Points.

Regular and Irregular Singular Points of a 2nd Order ODE (in Standard Form)

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \ldots. \quad (*)$$

The Point $\boldsymbol{x} = \boldsymbol{x_0}$ is called a ***Regular Singular Point*** of the ODE (*) if

(a) $\boldsymbol{x_0}$ is a Singular Point of the ODE (*)

(b) Both functions

$$A(x) = (\boldsymbol{x} - \boldsymbol{x_0})\,P(x)$$
$$B(x) = (\boldsymbol{x} - \boldsymbol{x_0})^2\,Q(x)$$

are analytic at $\boldsymbol{x} = \boldsymbol{x_0}$.

Otherwise the point $\boldsymbol{x_0}$ is called an ***Irregular Singular Point***

Example

$$y'' + [1/x^2(x+1)^2]y' + [x/(x^2-1)^2]\,y = 0 \qquad (2)$$

1. $\boldsymbol{x} = 1$ is a Regular Singular Points of ODE (2)
2. $\boldsymbol{x} = 0, -1$ are the Irregular Singular Points of the ODE (2).

I. Series Solution about Ordinary Points (O P)

Existence of Power Series Solution

Given

i. $x = x_0$ is an <u>O P</u> of the 2nd Order LDE:

$$y'' + P(x)\,y' + Q(x)\,y = 0 \;\ldots.\; (*)$$

ii. R is the distance from x_0 to the nearest singular point (Real or Complex) of the DE (*) in the Complex Plane.

Conclusion

i. We can find <u>Two LI Solutions</u> of (*) in the form of a PS centered at x_0 :

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

ii. The series will converge at least in the interval $| x - x_0 | < R$.

Method of Solution when $x_0 = 0$ is Ordinary Point

1. Find the Singular Points and the Radius of Convergence
2. Write $y(x) = \sum_{n=0}^{\infty} c_n x^n$
3. Find the Series for y' & y''
4. Substitute in the ODE.
5. Shift the Indices, if required, and compose the coefficients of same powers of x.
6. Equate the Coeff. of the powers of x to 0.
7. Find the Recurrence Relation.
8. Find 2 Linearly Independent Solutions

Exercises

[Find Rad. of Conv. of Solution Series]

1. Show that $(1+x^2)\,y'' + 3x\,y' + y = 0$ has two LI Series Solutions centered at Zero.

2. Find 2 LI Series Solutions in Powers of x to

a) $y'' - 2x\,y' - 4y = 0$

b) $y'' + (1+x^2)\; y' - 8xy = 0$

II. Series Solution about Regular Singular Points (R S P)

Method of Frobenius

Given

i. $x = x_0$ is a <u>R S P</u> of the 2nd Order LDE:

$$y'' + P(x)\,y' + Q(x)\,y = 0 \;\ldots.\; (\#)$$

ii. R is the distance from x_0 to the nearest singular point (Real or Complex) of the DE (*) in the Complex Plane.

Conclusion

i. There exists at least one Solution of (#) in the form of Series <u>with some constant r</u>:

$$y(x;r) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \;\ldots\ldots\; (**)$$

ii. The series will converge at least in the annulus $0 < | x - x_0 | < R$.

A Note on the Constant r in (**)

"Indicial Equation"

[For Simplicity we consider the RSP $x_0 = 0$]

1. The constant r in (**) is called the <u>Exponent of Singularity</u>.

2. Set $A(x) = xP(x)$; $B(x) = x^2 h(x)$.

$\Rightarrow A(x)$ & $B(x)$ will be analytic at $x_0 = 0$

$$\Rightarrow \quad A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n$$

3. We can write Eq. **(#)** as

$$x^2 y'' + x\,A(x)\,y' + B(x)\,y = 0 \;\ldots.\; (\#\#)$$

4. **Note:** Eq. **(##)** becomes Cauchy-Euler Eq. if $A(x)$ & $B(x)$ are constants. In this case, the Eq. **(##)** has a solution of the form $y = x^{\mathrm{m}}$ Here, m is a root of "$m\,(m - 1) + c\,m + d = 0$". It may be –ive or a fraction. In such case, the series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ will not match with $y = x^{\mathrm{m}}$.

5. **Remedy**: To include these possibilities in the series solution, we introduce additional parameter r in the proposed series and try $y(x;r) = \sum_{n=0}^{\infty} c_n x^{n+r}$.

6. This substitution in **(##)** after eliminating x^{r}, provides us the following Eq. corresponding to constant term:

$$r\,(r - 1) + A_0\, r + B_0 = 0; \;\; A_0 = A(x_0),\; B_0 = B(x_0)$$

This Eq. is known as "**Indicial Eq.**". Its solutions help us overcome the difficulty pointed in (4).

Method for Series Solution About R S P $x_0 = 0$

Part I: Roots of Indicial Eq.

Step 1: Check that $\mathbf{x_0 = 0}$ is an R S P.

Step 2: Find the Indicial Equation.

(a) Write the DE in the Form:

$$x^2 y'' + x A(x)\, y' + B(x)\, y = 0$$

(b) The Indicial Equation is:

$r(r-1) + A_0 r + B_0 = 0$;

where $A_0 = A(x_0)$, $B_0 = B(x_0)$

Step 3: Find the Roots r_1 & r_2 of the Indicial Equation. ($r_1 \geq r_2$)

Part II : Nature of Roots r_1 & r_2

3 cases for $r_1 - r_2$

$\neq$ Integer | $=$ +ive Integer | $= 0$

Series Solution

Part III : Case (i) $r_1 - r_2 \neq$ Integer

i. There will be two LI solutions of the Form:

$$y(x; r_1) = \sum_{n=0}^{\infty} c_n x^{n+r_1}\ ; \quad y(x; r_2) = \sum_{n=0}^{\infty} d_n x^{n+r_2}\ ;$$

i. Find for each series the constants c_n and d_n separately following the method of Series Solution about Ordinary Points.

Part III : Case (ii) $r_1 - r_2$ = +ive Integer

i. One Solution will be of the form

$$y(x; r_1) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

ii. 2nd Solution (Using Reduction of Order)

$$y_2(x) = u(x)\, y(x; r_1)$$

where

$$u(x) = \int \frac{e^{-\int [A(x)/x]dx}}{[y(x; r_1)]^2}\, dx$$

Note: In some cases, the solution corresponding to a smaller root provides two LI PS Solytions.

Important Steps for Finding of $u(x)$

Step 1: Evaluate $G(x) = e^{-\int [A(x)/x]dx}$

Step 2: Find the Power Series of $G(x)$:

If we know the Power Series of $A(x)$ then (Using Term by Term Integration)

$$\int [A(x)/x]\, dx = A_0 \ln x + A^*(x)$$

where: $$A^*(x) = \sum_{j=1}^{\infty} \frac{A_j}{j} x^j$$

[A_j = Coefficients in the Power Series of $A(x)$]

Step 3: Find a Few Terms the Power Series of $[y(x; r_1)]^2$

Step 4: Use Long Division to find a Few Terms the Power Series of $1/[y(x; r_1)]^2$

Step 5: Find the Power Series of the Product of the two Power Series obtained in Step 2 and Step 4. [Multiply the terms of the both PS]

Step 6: Use Term by Term Integration for the Power Series obtained in Step 5. This gives us the Power Series of $u(x)$.

Part III : Case (iii) $r_1 - r_2 = 0$

Here, the method for finding the 2 LI Solutions is exactly identical to that followed in Part III: Case (ii).

General Form of the 2nd LI Solution

Case (ii)

$$y_2 = C y(x; r_1) \ln x + \sum_{n=0}^{\infty} d_n x^{n+r_2}$$

[Here, the Constant C may be **0,** but $d_0 \neq 0$]

Case (iii)

$$y_2 = y(x; r_1) \ln x + \sum_{n=1}^{\infty} d_n x^{n+r_2}$$