Review of Power Series (I)

Infinite Series of Constants:

$$\sum_{n=k}^{\infty} c_n = c_k + c_{k+1} + \cdots + c_n \cdots\cdots$$

n = Index of the series (Dummy Variable)
k : A fixed integer (e.g. -16, 0, 1, 40,…)
c_k = 1st Term of the Series
c_n = nth (*or General*) Term of the Series

Shift of Index of Summation "n"

(Making a suitable substitution)

$$\sum_{n=k}^{\infty} c_n = \sum_{m=0}^{\infty} c_{m+k} \qquad [n = m + k]$$

$$\sum_{n=k}^{\infty} c_n = \sum_{j=1}^{\infty} c_{j+k-1} \qquad [n = m + k - 1]$$

Power Series in $(x - a)$ or centered at a

$$\sum_{n=0}^{\infty} c_n (x-a)^n \cdots\cdots \quad \text{(I)}$$

Here, c_n and a are Constants.
Note: For any value of x, a Power Series is a Series of Constants

Examples:

i. $\sum_{n=0}^{\infty} 3^{-n}(x-5)^n$ is a power series <u>Centered</u> at $x=5$

ii. Replace $x-5$ by u in (i). Then $\sum_{n=0}^{\infty} 3^{-n} n u^n$ is a power series in u centered at $u=0$.

iii. $\sum_{n=0}^{\infty} n 3^n$ is a <u>Series of Constants</u> obtained from (i) for $x=16$.

Convergence of Power Series [PS]

We say that the PS $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at $x = x_1$ if the Series of Constants $\sum_{n=0}^{\infty} c_n (x_1 - a)^n$ converges, e.g.

Example: The PS $\sum_{n=0}^{\infty} x^n$ converges for $x = 0.5$ but diverges for $x=1$.

Interval of Convergence

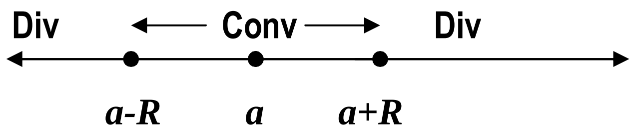
The set of all points for which a PS converges is called the Interval of Convergences of the PS

Example: The PS $\sum_{n=0}^{\infty} (x-1)^n$ converges for $0 < x < 2$ and diverges outside (0 , 2). The Interval of Convergence for this series is (0 , 2) .

Basic Convergence Theorem for PS

For a PS $\sum_{n=0}^{\infty} c_n (x-a)^n$, ONLY one of the following statements is True:

1. The PS Converges only at $x=a$.
2. The PS Converges for all Real x.
3. There is a +ive real R such that the PS converges for $|x-a|<R$, *i.e.* $a - R < x < a + R$ and Diverges for $|x - a| > R$.



Radius of Convergence of PS

The Number R appearing in Possibility (3) is called the Radius of Convergence of the PS.

Example: The Radius of Convergence of the PS $\sum_{n=0}^{\infty} (x-1)^n$ is 1.

Convergence at the End Points

If a PS $\sum_{n=0}^{\infty} c_n (x-a)^n$, converges in the Interval $(a-R\ ,\ a+R)$, it may or may not converge at the end Points $a \pm R$. Therefore, the convergence at the End Points is tested separately in order to decide about the Interval of Convergence.

Example: The Interval of Convergence of the PS $\sum_{n=0}^{\infty} (x-1)^n$ is (0,2). [The PS Div. at x=0,2.]

Ratio Test for Finding Radius of Convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$ …… (1)

Suppose that $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L,$

i. R= 1 / L is Radius of Convergence of PS (1)

ii. L = 0 $\Rightarrow$ The PS Converges for all Real x.

iii. L = ∞ $\Rightarrow$ The PS Converges for $x = a$.

Example

Find the Radius and Interval of Convergence of the PS $\sum_{n=0}^{\infty} \frac{n^2}{3^n}(x-4)^n$ **.**

Solution: Here, $c_n = n^2/3^n$, $a = 4$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{3^{n+1}}\cdot\frac{3^n}{n^2} = \frac{1}{3} \Rightarrow L = \frac{1}{3}$$

i. R = 3 is the Radius of Conv. of the PS.

ii. The PS Converges in (1 , 7)

iii The PS Diverges at x = 1, 7 (!!)

iv. Interval of Conv. of the PS = (1 , 7) .

Power Series as a Function

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \mathrm{F}(x) \text{ ……….. (2)}$$

Examples

(1) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$ (2) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n!} = \cos x$

(3) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (4) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$

i. The Domain of F(x) = Interval of Convergence of PS

ii. If R = Radius of Convergence of the PS in (2), then

a) F(x) is Continuous, Differentiable and Integrable on the interval (R - a , R + a)

b) F′(x) and ∫ F(x) dx can be found respectively by Term-by-Term Differentiation and Integration of PS(2).

Algebra of Power Series

i. Equality of 2 Series

$$\sum_{n=0}^{\infty} b_n (x-a)^n = \sum_{n=0}^{\infty} c_n (x-a)^n$$
$$\Leftrightarrow b_n = c_n \text{ for all n.}$$

Note: $\sum_{n=0}^{\infty} b_n (x-a)^n = 0 \Leftrightarrow b_n = 0$ for all n

ii. (Addition & Subtraction of 2 PS)

Write $\sum_{n=4}^{\infty} 2c_n x^{n-2} + \sum_{n=0}^{\infty}(n+1)c_n x^{n+1}$ **as One Series.**

Method

While adding two SeriesBoth Series must start with the Same Power of x.
[For this, Shift the Indices of Summation]
Coefficients of Same Power of x will be added.

Solution

i. Put $m = n$-2 in the 1st Series. Then

$$\sum_{n=4}^{\infty} 2c_n x^{n-2} = \sum_{m=2}^{\infty} 2c_{m+2} x^m$$

ii. Put $m = n$+1 in the 2nd Series. Then

$$\sum_{n=0}^{\infty}(n+1)c_n x^{n+1} = \sum_{m=1}^{\infty} mc_{m-1} x^m$$

iii. Add the Series

$$\sum_{n=4}^{\infty} 2c_n x^{n-2} + \sum_{n=0}^{\infty}(n+1)c_n x^{n+1}$$
$$= \sum_{m=2}^{\infty} 2c_{m+2} x^m + \sum_{m=1}^{\infty} mc_{m-1} x^m$$
$$= \sum_{m=2}^{\infty} 2c_{m+2} x^m + \left[c_0 + \sum_{m=2}^{\infty} mc_{m-1} x^m\right]$$
$$= c_0 + \sum_{m=2}^{\infty}[2c_{m+2} + mc_{m-1}]x^m$$

Recurrence Relation

$$2c_{m+2} + mc_{m-1} = 0$$

Multiplication of Series by a Series

Find First Four terms of the Power Series in x for the function $f(x) = e^x \ln(1-x)$

Method

1. Write the Power Series of both functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}; \quad \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

2. Expand both Series and Multiply term by term:

$$e^x \ln(1-x)$$
$$= -\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right]\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \cdots\right]$$
$$= -\left[x + \left(1+\frac{1}{2}\right)x^2 + \left(\frac{1}{3}+\frac{1}{2}+\frac{1}{2!}\right)x^3 + \left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2(2!)}\right)x^4 + \cdots\right]$$

(After gathering alike powers of x.)

Note:

Sometimes it is quite hard to find a solution of a simple ODE like

$$y'' + e^x y = 0$$

by the Known Integral / Substitution Methods. Solution for this type of Problems may be found in the form of Power Series.

What is Power Series Solution of a Differential Equation?

A series of the form $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$

"with appropriate choice of constants c_n and r" when satisfies a given DE is known as "**Power Series Solution of the DE**".

Example 1

i. Solve the ODE : $y' + 2y = 0$ ….. (1) **using a method of solutions for Ist Order ODE.**

ii. Solve the ODE (1) using the Power Series Solution.

(i) Solution: (1) is Separable Equation

$$dy/y = -2\,dx$$

$$\Rightarrow y = c\, e^{-2x}$$ is a Solution of (1)

(ii) Solution in Series for

$$y' + 2y = 0 \quad \ldots.. (1)$$

i. **Set** $y = \sum_{n=0}^{\infty} c_n x^n$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

ii. **Eq. (1) becomes**

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \ldots. (2)$$

iii. **Combine the similar Powers of x:**

[set $m = n$-1 in the 1st Series]

$$\sum_{m=0}^{\infty} (m+1) c_{m+1}\, x^m$$

iii. **Substituting in Eq (2) gives us:**

$$\sum_{m=0}^{\infty} [(m+1) c_{m+1}\, x^m + 2 c_m] x^m = 0$$

iv. By Identity Principle, each Coefficient of the Series is Zero:

$$[(m+1) c_{m+1} + 2 c_m] = 0$$

v. **Recurrence Relation**

$$c_{m+1} = \frac{-2 c_m}{m+1}; \qquad m \geq 0 \quad (*)$$

vi Calculation Parameters:

m	**C_m**	**Using (*)**
0	**C_1**	**$-2\, C_0 / 1$**
1	**C_2**	**$-2\, C_1/2 = 2^2\, C_0 / 2!$**
2	**C_3**	**$-2\, C_2 /3 = -\, 2^3\, C_0 /1.2.3 = -\, 2^3\, C_0 / 3!$**
.	**.**	
.	**.**	
n	**C_n**	**$(-1)^n\, 2^n\, C_0 / n!$**

v. The Solution in Series is given by

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n c_0}{n!} x^n$$

$$c_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = c_0 e^{-2x}$$