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Series Solution about R S P

Part (1) Difference of the Roots of Indicial Equation is Not an Integer

Use the Method of Frobenius to solve $2x^2y'' + xy' - (1+x)y = 0$

Step 1: Write the Equation in the Form:

$$y'' + P(x)y' + Q(x)y = 0$$

Hence, $x = 0$ is a regular singular point for the given DE.
$\Rightarrow$ we can find at least one series solution of the form

$$y(x;r) = \sum_{k=0}^{\infty} c_n x^{n+r}$$. Therefore,

Step 2 (*)

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \Rightarrow -(1+x)y = -\sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1}$$

n+r

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \qquad \Rightarrow xy' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \Rightarrow 2x^2y'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r}$$

Step 3: Change of Power is Required only in one Series, i.e. (*):
[$k=n+1$]

Step 4: Substitution into the DE gives:

$$2(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)c_k x^{k+r} - \sum_{k=0}^{\infty} c_k x^{k+r} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r} = 0$$

Step 5: Separate the terms for k = 0 in the 1st 3 Summations. Then Combine All

$$[2r(r-1) + r - 1]c_0 + \sum_{k=1}^{\infty} \{[2(k+r)(k+r-1) + (k+r) - 1]c_k - c_{k-1}\}x^{k+r} = 0$$

Note: $2r(r-1) + r - 1 = 0$ is the Indicial Equation & $c_0 \neq 0$
Step 6: Recurrence Relation:

$$(k+r-1)(2k+2r+1)c_k - c_{k-1} = 0 \qquad (**)$$

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Case 1: $r = 1$ in (**)

$$c_k = \frac{1}{k(2k+3)}c_{k-1}, \quad k = 1,2,3...$$

$k=1:\ c_1 = (1/1.5)c_0$

$k=2:\ c_2 = (1/2.7)c_1 = (1/[2!5.7])c_0$

$k=3:\ c_3 = (1/3.9)c_2 = (1/[3!5.7.9])c_0$

$k=4:\ c_4 = (1/4.11)c_3 = (1/[4!5.7.9.11])c_0$

⋮

Therefore, the Frobenius Series Solution is:

$$y(x) = x\{c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\}$$

$$= c_0x[1 + \frac{x}{5} + \frac{x^2}{2!5.7} + \frac{x^3}{3!5.7.9} + \frac{x^4}{4!5.7.9.11} + \cdots]$$

First Solution is given by:

$$y_1(x) = x + \frac{x^2}{5} + \frac{x^3}{2!5.7} + \frac{x^4}{3!5.7.9} + \frac{x^5}{4!5.7.9.11} +$$

Case 2: $r =$ -½ in (**)

$$c_k = \frac{1}{k(2k-3)}c_{k-1}, \quad k = 1,2,3..$$

$k=1:\ c_1 = -c_0$

$k=2:\ c_2 = (1/2.1)c_1 = (-1/[2!])c_0$

$k=3:\ c_3 = (1/3.3)c_2 = (-1/[3!1.3])c_0$

$k=4:\ c_4 = (1/4.5)c_3 = (-1/[4!1.3.5])c_0$

⋮

Therefore, the Frobenius Series Solution is

$$y(x) = x^{-1/2}\{c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\}$$

$$= c_0x^{-1/2}[1 - x - \frac{x^2}{2!} + \frac{x^3}{3!1.3} - \frac{x^4}{4!1.3.5} + \cdots]$$

Second LI Solution is given by:

$$y_2(x) = x^{-1/2}[1 - x - \frac{x^2}{2!} + \frac{x^3}{3!1.3} - \frac{x^4}{4!1.3.5} +$$

General Solution:

$$y(x) = a_1y_1(x) + a_2y_2(x)$$