King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
(Term 062)
Math 202-7,11 & 12
Major Exam 2

Time: 1:30 hours

Name: M XXX XXX [Student]

ID #:

Serial #: Sec #: 7

No Calculator is Allowed in this Exam
Show All Necessary Work

1	10 /10
2	10 /10
3	15 /15
4	30 /30
5	25 /25
6	10 /10
Total	100 /100

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO

(1) A thermometer is taken from an inside room to outside, where the air temperature is 5° F. After 1 minute the thermometer reads 55° F, and after 5 minutes it reads 30° F. What is the initial temperature of the inside room? (10pts)

$T_m = 5°F$

$T(1) = 55°F$

$T(5) = 30°F$

$T(0) = ?$

$T - T_m = ce^{kt}$

$T - 5 = ce^{kt}$

$T(1) = 55°F$

$55 - 5 = ce^{k}$

$50 = ce^{k}$

$c = \frac{50}{e^k} \quad \cdots (1)$

$T(5) = 30$

$30 - 5 = ce^{5k}$

$25 = ce^{5k} \quad \cdots (2)$

Sub. (1) in (2) :-

$25 = \frac{50}{e^k} e^{5k}$

$25 = 50 e^{4k}$

$e^{4k} = \frac{25}{50} = \frac{1}{2}$

$4k = \ln\left(\frac{1}{2}\right)$

$\boxed{k = \frac{\ln\left(\frac{1}{2}\right)}{4}}$

$\therefore c = \frac{50}{e^{\frac{1}{4}\ln\left(\frac{1}{2}\right)}} = \frac{50}{e^{\ln(0.5)^{0.25}}} = \frac{50}{(0.5)^{0.25}} = \frac{50}{(0.5)^{0.25}}$

$\left(\frac{1}{2}\right)^{\frac{1}{4}} = \frac{1^{\frac{1}{4}}}{2^{\frac{1}{4}}} = \frac{1}{(2)^{\frac{1}{4}}}$

$\therefore T - 5 = \frac{50}{(0.5)^{0.25}} e^{\frac{1}{4}\ln\left(\frac{1}{2}\right)t}$

$T(0) \Rightarrow T - 5 = \frac{50}{(0.5)^{0.25}} e^{0}$

$\boxed{T_{(0)} = \left(\frac{50}{(0.5)^{0.25}} + 5\right)° F} \rightarrow$ Ans.

$\therefore T_{(0)} = 59.5 + 5$

$\boxed{T_{(0)} = 64.5 °F} \rightarrow$ Ans.

(2) Prove that the functions

$(C_2+3C_3)\csc^2 x=0$

$C_2+3C_3=0 \rightarrow C_2=-3C_3$

$$f_1(x)=5,\ f_2(x)=\cot^2 x,\ f_3(x)=3\csc^2 x \text{ and } f_4(x)=x^4 \quad \boxed{C_3=\frac{-5}{3}}$$

are linearly dependent on the interval $(0,\pi)$. (10pts)

→ if we can express one function in terms of ~~others~~ linear combination of others &

→ So L.dep.

$f_1(x)=C_1 f_2(x)+C_2 f_3(x)+C_3 f_4(x)$

$\boxed{\text{or by } W}$

$5=C_1\cot^2 x+3C_2\csc^2 x+C_3 x^4$

$5=C_1(\csc^2 x-1)+3C_2\csc^2 x+C_3 x^4$

$5=C_1\csc^2 x-C_1+3C_2\csc^2 x+C_3 x^4$

$=(C_1+3C_2)\csc^2 x+C_3 x^4-C_1$

$\sin^2 x+\cos^2 x=1$

$1+\cot^2 x=\csc^2 x$

$\cot^2 x=\csc^2 x-1$

put $C_3=0$

$-C_1=5$

$\boxed{C_1=-5}$

$C_1+3C_2=0$

$-5+3C_2=0$

$3C_2=5$

$\boxed{C_2=\frac{5}{3}}$

$1-\sin^2 x+1$

$2-\sin^2 x$

$\therefore 5=(-5+5)\csc^2 x+0x^4+5$

$5=5$ -- (L) has been proved.

Since Not all constant equals zero & that we could express $f_1(x)$ in terms of linear combinations of other functions, so Linearly dependent on given interval.

...coefficients method, find the most suitable form of y_p for

$$(D^2 - 2D + 2)y = x^2 e^x \sin x - 2xe^x \cos x.$$

Do not evaluate the constants.

(15pts)

$(D^2-2D+2)^3$ (- -)

first find y_h :-

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\therefore y_h = c_1 e^x \cos x + c_2 e^x \sin x$

To find y_p :-

$(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$

Annihilator of $x^2 e^x \sin x - 2xe^x \cos x = (D^2 - 2D + 2)^3$

$(\lambda^2 - 2\lambda + 2)^3 (\lambda^2 - 2\lambda + 2) = 0 \Rightarrow (\lambda^2 - 2\lambda + 2)^4 = 0$

$\therefore \lambda = 1 \pm i$ repeated four times :- (x^3)

$\therefore y = \underbrace{c_1 e^x \cos x + c_2 e^x \sin x}_{y_h} + \underbrace{c_3 e^x x \cos x + c_4 e^x x \sin x + c_5 e^x x^2 \cos x + c_6 e^x x^2 \sin x + c_7 e^x x^3 \cos x + c_8 e^x x^3 \sin x}_{y_p}$

$y_p = c_3 e^x x \cos x + c_4 e^x x \sin x + c_5 e^x x^2 \cos x + c_6 e^x x^2 \sin x + c_7 e^x x^3 \cos x + c_8 e^x x^3 \sin x$

$y_p = e^x x (c_3 \cos x + c_4 \sin x) + e^x x^2 (c_5 \cos x + c_6 \sin x) + e^x x^3 (c_7 \cos x + c_8 \sin x)$

Ans.

(4) Solve each of the following differential equations. (30pts)

(i) $y'' - 2y' + y = x^2 e^x$ (by variation of parameters)

Ans $(\lambda - 1)^2$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1, 1$$

$$\therefore y_h = c_1 \underset{\to y_1}{e^x} + c_2 \underset{\to y_2}{x e^x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + e^x x \end{vmatrix} = e^x(e^x + e^x x) - xe^{2x}$$

$$= e^{2x} + e^{2x}x - xe^{2x} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ x^2 e^x & e^x + e^x x \end{vmatrix} = 0 - xe^x(x^2 e^x) = -x^3 e^{2x}.$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & x^2 e^x \end{vmatrix} = e^{2x} x^2$$

$$\therefore u_1' = \frac{W_1}{W} = \frac{-x^3 \cancel{e^{2x}}}{\cancel{e^{2x}}} = -x^3, \qquad \therefore u_1 = \int -x^3\,dx = -\frac{x^4}{4}.$$

$$u_2' = \frac{W_2}{W} = \frac{e^{2x} x^2}{e^{2x}} = x^2, \qquad \therefore u_2 = \int x^2\,dx = \frac{x^3}{3}.$$

$$\therefore y_p = -\frac{x^4}{4}e^x + \frac{x^3}{3}xe^x = -\frac{x^4}{4}e^x + \frac{x^4}{3}e^x$$

general sol. is:

$$\therefore y = y_h + y_p$$

$$\boxed{y = c_1 e^x + c_2 x e^x - \frac{x^4}{4}e^x + \frac{x^4}{3}e^x} \to \text{Ans.}$$

add

(ii) $x^2y'' - xy' + y = 0$.

Cauchy - Euler.

the form of Cauchy - Euler Equation :-

let $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)\ x^{m-2}$

Sub. in given D.E. :-

$\cancel{x^2}\ m(m-1)\cancel{x^{m-2}} - \cancel{x}\ m\cancel{x^{m-1}} + \cancel{x^m} = 0$

$m(m-1) - m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$m^2 + (b-a)m + c = 0$

$-1 - 1$

$m^2 - 2m + 1 = 0$

$(\ln x)^2$

$(\ln x)^3$

$\therefore\ y = c_1 x^1 + c_2 x^1 \ln x$

$\boxed{y = c_1 x + c_2 x \ln x} \rightarrow$ Ans.

...tion of $y^{(4)} - k^2 y'' = 1$. (15

find y_h :-

$$\lambda^4 - k^2\lambda^2 = 0$$

$$\lambda^2(\lambda^2 - k^2) = 0$$

$$\lambda = 0, 0$$

$$\lambda = \pm\sqrt{k^2} = \pm k$$

$$\therefore \boxed{y_h = c_1 + c_2 x + c_3 e^{+kx} + c_4 e^{-kx}}$$

find y_h

$$\lambda^4 - k^2\lambda^2 = 0$$

$$\lambda^2(\lambda^2 - k^2) = 0$$

$$\lambda = 0, 0$$

$$\lambda^2 = k^2$$

$$\lambda = \pm\sqrt{k^2}$$

$$\lambda = k$$

find y_p by Using <u>Undetermind coefficient</u> :-

Annilihater of 1 is $\underline{D}$.

$$\therefore \lambda(\lambda^4 - k^2\lambda^2) = 0$$

$$\lambda(\lambda^2)(\lambda^2 - k^2) = 0$$

$$\lambda = 0, 0, 0 \qquad \lambda = \pm k$$

$$\therefore y = \underbrace{\underbrace{c_1 + c_2 x} + c_3 x^2 + \underbrace{c_4 e^{kx} + c_5 e^{-kx}}}_{y_h}$$

$$\therefore y_p = c_3 x^2 = Ax^2$$

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p''' = 0$$

$$y_p'''' = 0$$

$$\therefore 0 - k^2(2A) = 1$$

$$\therefore -2Ak^2 = 1$$

$$\therefore y_p = \frac{x^2}{-2k^2}$$

$\therefore$ general sol =

$$\boxed{y = y_h + y_p = c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} + \frac{x^2}{-2k^2}}$$

Ans.

(b) The general solution to the non-homogeous differential equation $y^{(4)} - k^2 y'' = q(x)$ can be written as $y(x) = c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} + \frac{1}{k^2}\int q(x)x\,dx - \frac{x}{k^2}\int q(x)dx + \frac{e^{kx}}{2k^3}\int q(x)e^{-kx}dx - \frac{e^{-kx}}{2k^3}\int q(x)e^{kx}dx$. (*)
Show that the general solution computed in part (a) match the general solution (*).

(10pts)

By Superposition principle

$$y^{(4)} - k^2 y'' = q(x)$$

$$q(x) = 1x^0 = 1$$

So

$$y(x) = c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} + \frac{1}{k^2}\underbrace{\int x\,dx}_{\frac{x^2}{2}} - \frac{x}{k^2}\underbrace{\int dx}_{x} + \frac{e^{kx}}{2k^3}\int e^{-kx}dx - \frac{e^{-kx}}{2k^3}\int e^{kx}dx$$

$$= c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} + \frac{x^2}{2k^2} - \frac{x^2}{k^2} + \frac{e^{kx}}{2k^3}\cdot\frac{1}{-k}e^{-kx} - \frac{e^{-kx}}{2k^3}\frac{1}{k}e^{kx}$$

$$= c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} - \frac{x^2}{2k^2} - \frac{1}{2k^4} - \frac{1}{2k^4}$$

$$= \underbrace{c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} - \frac{x^2}{2k^2} - \frac{1}{k^4}}$$, since k is constant

$c_1 + \frac{(-1)}{k^4} = C_1$ (all constants so we can added ((combine them))).

$$\therefore\ y = \underbrace{C_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx}}_{y_h} - \underbrace{\frac{x^2}{2k^2}}_{y_p}$$

→ as general solution that we found in (a) ((Match it)).

$\int e^{-5x}$

$-\frac{1}{5}e^{-5x}$

$= e^{-5x}$

(6) We have shown in class that if $x = e^t$, then

$$x\frac{dy}{dx} = \frac{dy}{dt} \text{ and } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Show that $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$. (10pts)

let $x = e^t$

$t = \ln x$

$dt = \frac{1}{x}\,dx$

$\frac{dt}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\underbrace{\frac{d}{dx}}_{\frac{d}{dt}\frac{dt}{dx}}\frac{dy}{dt} = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dt} + \frac{-1}{x^2}\underbrace{\frac{d}{dx}}_{\frac{d}{dt}\frac{dt}{dx}\ \frac{1}{x}}\frac{dy}{dt} + \frac{-2}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^2}\underbrace{\frac{d}{dx}}_{\frac{d}{dt}\frac{dt}{dx}\ \frac{1}{x}}\frac{d^2y}{dt^2}$$

$-x^{-2}$

$\to$

$2x^{-3}$

$\frac{2}{x^3}$

$$= \frac{2}{x^3}\frac{dy}{dt} - \frac{1}{x^3}\frac{d^2y}{dt^2} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^3}\frac{d^3y}{dt^3}$$

Sub. in DE.

Dr. M. R. Alfuraidan

$$x^3\frac{d^3y}{dx^3} = x^3\left[\left(\frac{2}{x^3}\frac{dy}{dt}\right) - \frac{1}{x^3}\frac{d^2y}{dt^2} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^3}\frac{d^3y}{dt^3}\right)$$

$$= 2\frac{dy}{dt} - \frac{d^2y}{dt^2} - 2\frac{d^2y}{dt^2} + \frac{d^3y}{dt^3}$$

$$= \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = \text{R.H.S}$$

So verified as required shown

So $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$.