KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

MATH 202-(062)

First Major Exam

March 21, 2007

Time: 90 Minutes

I.D.: Name: Solution Sec.#

Show All Necessary Work

Question	Points
1	/10
2	/10
3	/12
4	/20
5	/24
6	/12
7	/12
Total	/100

Dr. Mohammad Samman

1. (a) For each of the following, state whether the equation is linear or nonlinear, and give its order.

Equation	Linearity	Order
$x(y'')^3 + (y')^4 - y = 0$	Non-linear	2
$(\cos x)y''' - (\sin x)y' - 2 = 0$	linear	3
$y' = 1 - xy + y^2$	Non-linear	1
$yy' + 2y + 3 - x^2 = 0$	Non-linear	1

$(y'')^3$
y^2
$y y'$

(b) Check the following 1^{st} order ODEs and write in the brackets S for *Separable*, L for *Linear*, E for *Exact*, H for *Homogeneous*, and B for *Bernoulli*:

i. $(y - xy^2)dy = ydx$ (L) in x

ii. $(e^{y/x} + e^{x^3/y^3} + 1)dy = (1 + \ln(y/x))dx$ (H)

iii. $x^y \ln x dy = -yx^{y-1}dx$ (S) L, E

iv. $u^2 + u = x^2 u'$ (S) B

v. $(x^2 y - y^3)dx = x^3 dy$ (H) B

vi. $(x^2 - 4x - 3xy + 4)dx = (x^2 - x - 2)dy$ (L)

2. (a) It is known that $y = \dfrac{2+2ce^{4x}}{1-ce^{4x}}$ is a one parameter family of Solutions of the ODE $y' = y^2 - 4$. Find a Singular Solution of this ODE.

Clearly, $y = -2$ satisfies the DE $y' = y^2 - 4$.

$\Rightarrow$ $y = -2$ is a solution to the given DE.

But we can not obtain this solution from the above one parameter family of solutions. i.e. there is no value for C which will satisfy $y = -2$. Hence the solution $y = -2$ is singular.

(b) Discuss the validity of the following statement:

The DE: $(9 - y^2)y' = x^2$ may have more than one solution in the region R in which $y < -3$.

$$y' = \frac{x^2}{9 - y^2} = f(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{2yx^2}{(9 - y^2)^2}$$

f and $\dfrac{\partial f}{\partial y}$ is continuous $\forall\, y < -3$

$\therefore$ the given DE must have one and only one solution in the specified region. i.e. the given statement is not correct.

3. Consider the autonomous first-order differential equation $\frac{dy}{dx} = y^2 + 2y - 8$

(a) Find the critical points of the given equation.

$$\frac{dy}{dx} = y^2+2y-8 = f(y)$$

$$f(y)=0 \Rightarrow y^2+2y-8 = (y-2)(y+4)=0$$

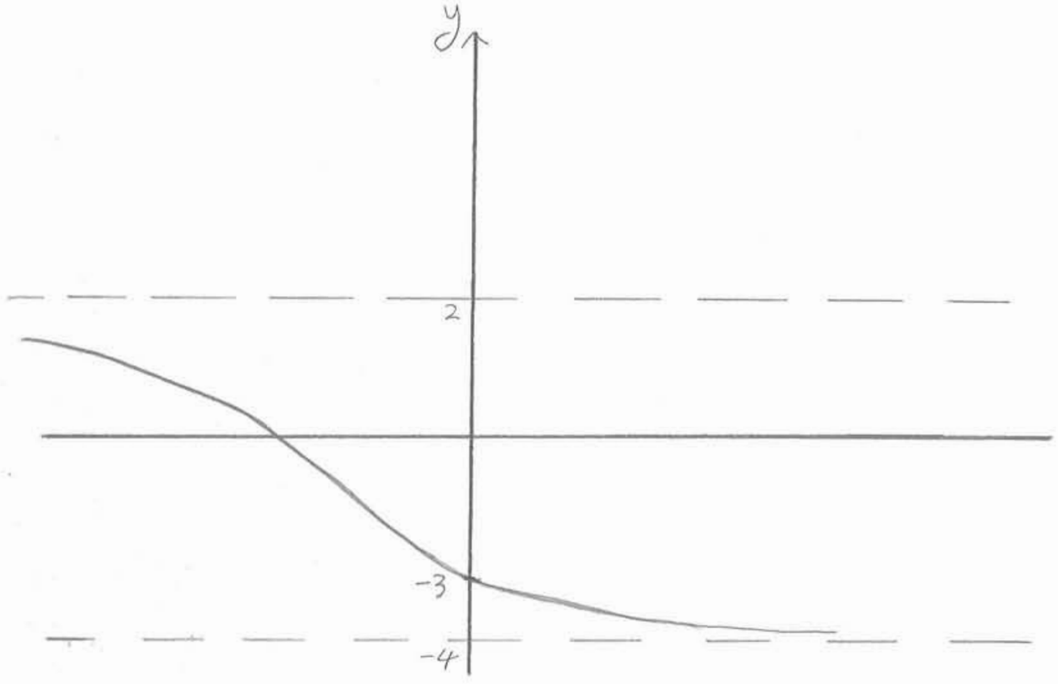
$$\Rightarrow \begin{cases} y = 2 \\ y = -4 \end{cases}$$ are critical points

(b) Discuss the stability at each point in (a).

$y=2$ is unstable (repeller)

$y=-4$ is asymptotically stable (attractor)

(c) Given the initial condition $y(0) = -3$, sketch the graph of the solution.



4. **(a)** Verify that the following D.E. is not exact, and then find the **integrating factor** that can be used in order to convert it to exact: **(Do not find** the solution)

$(1-x^2)dy = [xy + (1+x^2)x\ln x]dx$

$$\left[xy+(1+x^2)x\ln x\right]dx + (x^2-1)dy = 0$$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = x \quad , \quad \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Not exact.}$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x^2-1}\left[x-2x\right] = \frac{-x}{x^2-1}$$

$$\mu = e^{\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]dx} = e^{-\int \frac{x}{x^2-1}dx} = e^{-\frac{1}{2}\ln|x^2-1|} = (x^2-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2-1}}.$$

(b) Convert (**without solving**) the **Bernoulli D.E.** $x^2 dy = \left((\cos x)y - 2\sqrt{y}\ln x\right)dx$ into **linear D.E.**

$$\frac{dy}{dx} = \frac{(\cos x)y - 2\sqrt{y}\ln x}{x^2}$$

$$\frac{dy}{dx} - \left(\frac{\cos x}{x^2}\right)y = \frac{-2\ln x}{x^2}\, y^{\frac{1}{2}} \quad \text{(Bernoulli)}.$$

To Convert it into linear:

Put $w = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \Rightarrow y = w^2 \Rightarrow \frac{dy}{dx} = 2w\frac{dw}{dx}$

The above equation becomes:

$$2w\frac{dw}{dx} - \left(\frac{\cos x}{x^2}\right)w^2 = \frac{-2\ln x}{x^2}\,w$$

$$\frac{dw}{dx} - \left(\frac{\cos x}{2x^2}\right)w = -\frac{\ln x}{x^2} \quad \text{which is linear.}$$

5. Solve the following initial value problems:

(a) $(x^2+1)(y^2+4)y'+2=0, \qquad y(0)=0.$

$$(x^2+1)(y^2+4)\frac{dy}{dx} = -2$$

$$(y^2+4)\,dy = \frac{-2}{x^2+1}\,dx$$

Integrating:

$$\frac{y^3}{3}+4y = -2\tan^{-1}x + C$$

Using the initial Condition $y(0)=0 \implies C=0$

Hence, the solution is:

$$\frac{y^3}{3}+4y+2\tan^{-1}x = 0$$

(b) $y' = xy - x, \quad y(1)=2.$

$$\frac{dy}{dx} - xy = -x$$ which is linear. [also separable]

$$\mu = e^{\int -x\,dx} = e^{-x^2/2}$$

$$\frac{d}{dx}\left[e^{-x^2/2}y\right] = -xe^{-x^2/2}$$

$$e^{-x^2/2}y = \int -xe^{-x^2/2}\,dx = e^{-x^2/2}+C$$

i.e. $$e^{-x^2/2}y = e^{-x^2/2}+C$$

$$y = 1+Ce^{x^2/2}$$

Since $y(1)=2 \implies 2=1+Ce^{\frac{1}{2}} \implies Ce^{\frac{1}{2}}=1 \implies C=e^{-\frac{1}{2}}$

$\therefore$ the Solution is $y=1+e^{-\frac{1}{2}}e^{\frac{x^2}{2}}$ i.e. $y=1+e^{\frac{x^2-1}{2}}$.

6. Find the general solution of the DE: $y' = \dfrac{e^x \cos y - 2xy}{e^x \sin y + x^2}$

$$\frac{dy}{dx} = \frac{e^x \cos y - 2xy}{e^x \sin y + x^2}$$

$$\left(e^x \cos y - 2xy\right)dx + \left(-e^x \sin y - x^2\right)dy = 0$$

$$M\,dx + N\,dy = 0$$

$$M(x,y) = e^x \cos y - 2xy, \quad N(x,y) = -e^x \sin y - x^2$$

$$\frac{\partial M}{\partial y} = -e^x \sin y - 2x, \quad \frac{\partial N}{\partial x} = -e^x \sin y - 2x$$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ the given DE is Exact.

So, $\dfrac{\partial f}{\partial x} = M$, $\dfrac{\partial f}{\partial y} = N$

$$f(x,y) = \int M\,dx$$

$$= \int \left(e^x \cos y - 2xy\right)dx = e^x \cos y - x^2 y + g(y) \text{ --- } (*)$$

$$\frac{\partial f}{\partial y} = -e^x \sin y - x^2 + g'(y) = N = -e^x \sin y - x^2$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = C$$

Thus, $(*)$ becomes $f(x,y) = e^x \cos y - x^2 y + C$

Hence, the solution is

$$e^x \cos y - x^2 y + C = 0$$

7. Verify that the following D.E. is homogeneous and then solve it:

$(2y^2 + x^2)dx - xydy = 0$

$M\,dx + N\,dy = 0$

$M(x,y) = 2y^2 + x^2$, $N(x,y) = -xy$

$$M(tx,ty) = 2(ty)^2 + (tx)^2 = 2t^2y^2 + t^2x^2 = t^2(2y^2 + x^2) = t^2 M(x,y)$$

$$N(tx,ty) = -(tx)(ty) = -t^2xy = t^2(-xy) = t^2 N(x,y)$$

∴ Both M and N are homogeneous functions of degree 2.
∴ the given DE is homogeneous.

To Solve it, let $y = ux$
$dy = u\,dx + x\,du$

$$(2u^2x^2 + x^2)dx - ux^2(u\,dx + x\,du) = 0$$

$$2u^2x^2dx + x^2dx - u^2x^2dx - ux^3du = 0$$

$$u^2x^2dx + x^2dx - ux^3du = 0$$

$$x^2(u^2+1)dx - ux^3du = 0$$

$$\frac{dx}{x} - \frac{u}{u^2+1}du = 0 \Rightarrow \frac{dx}{x} = \frac{u}{u^2+1}du$$

$$\ln|x| = \tfrac{1}{2}\ln|u^2+1| + \ln|C|$$

$$= \ln|u^2+1|^{\frac{1}{2}} + \ln|C|$$

$$\ln|x| = \ln\left|\frac{y^2}{x^2}+1\right|^{\frac{1}{2}} + \ln|C| = \ln\left|C\left(\frac{x^2+y^2}{x^2}\right)^{\frac{1}{2}}\right|$$

$$x = C\sqrt{\frac{x^2+y^2}{x^2}}$$