Ex 4 Sketch the graph of $f(x) = \frac{x^3-x^2+4}{x-1}$

Sol.

* $D = \mathbb{R} \setminus \{1\} = (-\infty,1)\cup(1,\infty)$

* Intercepts: $x=0 \Rightarrow y=-4$ $\boxed{(0,-4)}$

  $y=0 \Rightarrow x^3-x^2+4=0$ [difficult to solve but we are sure that there is a zero between -2 and -1 (why?) and we can approximate it]

* No symmetry:

* Asymptotes: $\lim_{x\to 1^+} f(x) = +\infty$, $\lim_{x\to 1^-} f(x) = -\infty \Rightarrow$ $\boxed{x = 1 \text{ is a V. A.}}$

  Since $\deg(x^3-x^2+4) > 1+\deg(x-1) \Rightarrow f(x)$ has a curvilinear-asymptote, so

  as $\frac{x^3-x^2+4}{x-1} = x^2 + \frac{4}{x-1} \Rightarrow$ $\boxed{y = x^2 \text{ is a curvilinear asymptote}}$

* Increase-decrease: $f'(x) = \frac{(x-1)(3x^2-2x)-(x^3-x^2+4)}{(x-1)^2} = \frac{2x^3-4x^2+2x-4}{(x-1)^2}$

  $f'(x)=0 \Rightarrow 2x^3-4x^2+2x-4=0 \Rightarrow (x-2)(2x^2+2)=0 \Rightarrow$ $\boxed{x=2 \text{ critical pt}}$

* Extrema: Using 2nd derivative test:

  f': $-$ (1) $-$ 2 $+$

  $f''(x) = 2 + \frac{8}{(x-1)^3}$ (verify!) $\Rightarrow f''(2) = 10 > 0 \Rightarrow f(x)$ has $\boxed{\text{a relative min at } x=2}$

* Concavity $f''(x)=0 \Rightarrow 2+\frac{8}{(x-1)^3}=0 \Rightarrow 2 = \frac{-8}{(x-1)^3} \Rightarrow (x-1)^3=-4 \Rightarrow x-1=\sqrt[3]{-4}$

  $\Rightarrow x = 1+\sqrt[3]{-4} \approx -0.6$

  $\Rightarrow -0.6$ is an inflection point.

  f'': $+$ -0.6 $-$ (1) $+$

  $\cup$ $\cap$ $\cup$