Solution MATH 102 [Homework 5]

1. Find the local quadratic approximation of $f(x)=\cos^{-1} x$ about a=0, and approximate $\cos^{-1}(0.01)$.

i- $p_2(x) = f(0) + f'(0)\cdot x + \frac{1}{2!} f''(0)\, x^2$

ii-

$f(x) = \cos^{-1} x$	$f(0) = \cos^{-1} 0 = 1$
$f'(x) = \frac{-1}{\sqrt{1-x^2}}$	$f'(0) = -1$
$f''(x) = \frac{-x}{(1-x^2)^{3/2}}$	$f''(0) = 0$

(iii) $p_2(x) = 1 - x$

(iv) $\cos^{-1}(0.01) \approx p_2(0.01) = 1 - 0.01$

$= 0.99$

2. Find the n^{th} Taylor polynomial of $f(x) = \sin x$ about $x = \frac{\pi}{3}$.

i) $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\pi/3)}{k!} \left(x - \frac{\pi}{3}\right)^k$

(ii)
$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x$
$\vdots$
$f^{(7)}(x) = -\cos x$

- When $n = 0, 4, 8, \ldots$ $f^{(n)}(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- When $n = 1, 5, 9, \ldots$ $f^{(n)}(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$
- When $n = 2, 6, 10, \ldots$ $f^{(n)}(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}$
- When $n = 3, 7, 11, \ldots$ $f^{(n)}(\frac{\pi}{3}) = -\cos\frac{\pi}{3} = \frac{-1}{2}$

3. i. Find the n^{th} Maclaurin polynomial of $f(x) = \ln(1+x)$.

(i) $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

(ii) $f(x) = \ln(1+x)$ $\quad \boxed{f(0) = \ln 1 = 0}$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = (-1)(1+x)^{-2}$

$f'''(x) = (-1)(-2)(1+x)^{-3}$

$\vdots$

(*) $f^{(k)}(x) = (-1)(-2)\cdots(-(k-1))(1+x)^{-k} = (-1)^{k-1}((k-1)!)(1+x)^{-k}$

$f^{(k)}(0) = (-1)^{k-1}(k-1)!$ when $k \geq 1$

$p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} x^k$

$\Rightarrow \boxed{p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} x^k}$

ii. Using (i), approximate ln(1.1) to 3 decimal accuracy. [Use the interval [0,1]

1) Want to find $\boxed{n}$ so that $|R_n(0.1)| \leq .0005$

2) • Note $|R_n(x)| \leq \frac{M}{(n+1)!} |x^{n+1}|$

• Here $M = \max_{[0,1]} |f^{(n+1)}(x)| = \max_{[0,1]} |(-1)^n \, n! \, (1+x)^{-(n+1)}|$ (see (*) above)

• Note $(1+x)^{-(n+1)} = \frac{1}{(1+x)^{n+1}} \searrow$ on $[0,1]$

$\Rightarrow \max_{[0,1]} \left|\frac{1}{(1+x)^{n+1}}\right| = \frac{1}{(1+0)^{n+1}} = 1$

• $M = n!$

• $|R_n(x)| \leq \frac{n!}{(n+1)!} |x|^{n+1}$

$\Rightarrow |R_n(0.1)| \leq \frac{(0.1)^{n+1}}{n+1}$

• Find n so that $\frac{(0.1)^{n+1}}{n+1} \leq 0.0005$

Using Calculator: For $n=2$

$|R_2(0.1)| \leq (0.1)^3/3 \approx .0003$

(3) Therefore

$\ln(1.1) = f(0.1) \approx p_2(0.1) = \frac{(-1)^0}{1!}(0.1) + \frac{(-1)^1}{2!}(0.1)^2$

$= 0.1 + \frac{-1}{2}\left(\frac{1}{100}\right) = \boxed{0.0950}$

4. Find the general (n^{th}) term of the following sequences, and find the limit if it exists:

i. $\frac{1}{3}, \frac{3}{7}, \frac{5}{11}, \frac{7}{15}, \ldots;$

$a_1 = \frac{1}{3} = \frac{1}{1+2}$

$\boxed{a_2 = \frac{3}{7} = \frac{2(2)-1}{4(2)-1}}$

$a_3 = \frac{5}{11} = \frac{2(3)-1}{4(3)-1}$

$a_4 = \frac{7}{15} = \frac{2(4)-1}{4(4)-1}$

$\vdots$

$a_n = \frac{2n-1}{4n-1}, \quad n = 1, 2, 3, \ldots$

$\boxed{\text{Note: } \lim_{n\to\infty} a_n = \frac{2}{4} = \frac{1}{2}}$

ii. $(1-\frac{1}{2}), (\frac{1}{2}-\frac{1}{3}), (\frac{1}{3}-\frac{1}{4}), \ldots$

$a_1 = 1 - \frac{1}{2} = \frac{1}{1} - \frac{1}{1+1}$

$a_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{2} - \frac{1}{2+1}$

$a_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{3} - \frac{1}{3+1}$

$\vdots$

$a_n = \frac{1}{n} - \frac{1}{n+1}$

i.e.

$\boxed{a_n = \frac{1}{n} - \frac{1}{n+1}}$

$\boxed{\text{Note } \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0}$

5. Find the limit of the following sequences if exists:

i. $\left\{\left(\frac{n^2+5n-5}{n^2+6}\right)^n\right\}_{n=1}^{\infty}$

$y = \left(\frac{x^2+5x-5}{x^2+6}\right)^x$

$\Rightarrow \ln y = x \ln\left(\frac{x^2+5x-5}{x^2+6}\right)$

$(\infty \cdot 0 \text{ form})$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+5x-5}{x^2+6}\right)}{(1/x)}$

$\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to\infty} (\ln y)$

$= \lim_{x\to\infty} \left(\frac{x^2+6}{x^2+5x-5}\right) \cdot \frac{(x^2+6)(2x+5)-(x^2+5x-6)(2x)}{(x^2+6)^2} \cdot (-1/x^2)$

$= \lim_{x\to\infty} \frac{\text{Term in Highest Power of } x}{\text{Term in Highest Power of } x}$

$= \lim_{x\to\infty} \frac{x^2(-5x^2)}{x^4} = -5$

$\Rightarrow \lim_{x\to\infty} y = e^{-5}$

$\Rightarrow \lim_{n\to\infty} a_n = e^{-5}$

ii. $\left\{\sqrt{4n^6+5n^3} - 2n^3\right\}_{n=1}^{\infty}$

① $f(x) = \sqrt{4x^6+5x^3} - 2x^3 \qquad (\infty - \infty)$

$= \frac{\sqrt{4x^6+5x^3} - 2x^3}{\sqrt{4x^6+5x^3} + 2x^3} \left(\sqrt{4x^6+5x^3} + 2x^3\right)$

$= \frac{4x^6+5x^3-4x^6}{\sqrt{4x^6+5x^3} + 2x^3}$

② $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5x^3}{\sqrt{4x^6+5x^3} + 2x^3}$

$= \lim_{x\to\infty} \frac{5x^3/x^3}{\sqrt{4x^6/x^6 + 5x^3/x^6} + 2x^3/x^3}$

$= \frac{5}{2+2} = 5/4$

$\Rightarrow \boxed{\lim_{x\to\infty} a_n = 5/4}$

iii. $\left\{(-1)^n \frac{8n^4+9}{7n^4-6n}\right\}_{n=1}^{\infty}$.

i- $\lim_{\substack{n\to\infty \\ (n \text{ even})}} (-1)^n \frac{8n^4+9}{7n^4-6n} = \lim_{n\to\infty} \frac{8n^4+9}{7n^4-6n} = \frac{8}{7}$

ii- $\lim_{\substack{n\to\infty \\ (n \text{ odd})}} (-1)^n \frac{8n^4+9}{7n^4-6n} = \lim_{n\to\infty} \frac{-(8n^4+9)}{7n^4-6n} = -\frac{8}{7}$

iii- Ans. $\boxed{\lim_{n\to\infty} a_n \text{ D.N.E.}}$

6. **i. Page 657: Q. 31 (C):** Starting with n =1, and considering the even and odd terms separately, find a formula for the general term of the sequence:

$$1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \frac{1}{9}, \frac{1}{9}, \cdots$$

$a_1 = 1$ $\quad a_2 = \frac{1}{3}$

$a_3 = \frac{1}{3}$ $\quad a_4 = 1/5$

$a_5 = \frac{1}{5}$ $\quad a_6 = 1/7$

$\vdots$ $\quad \vdots$

$\boxed{a_{2n+1} = \frac{1}{2n+1}}$ $\quad \boxed{a_{2n} = \frac{1}{2n+1}}$

$$a_n = \begin{cases} 1/n, & n \text{ is odd} \\ 1/(n+1), & n \text{ is even} \end{cases}$$

ii. **Page 657: Q. 38 :** Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where

$$a_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3}$$

a. Find

$a_1 = \frac{1}{1^3} = 1$

$a_2 = \frac{1}{2^3} + \frac{2^2}{2^3} = \frac{1}{8} + \frac{1}{4} = \frac{5}{8}$

$a_3 = \frac{1}{3^3} + \frac{2^2}{3^3} + \frac{3^2}{3^3} = \frac{1+4+9}{3^3} = \frac{14}{27}$

$a_4 = \frac{1}{4^3} + \frac{2^2}{4^3} + \frac{3^2}{4^3} + \frac{4^2}{4^3} = \frac{1+4+9+16}{4^3} = \frac{30}{64}$

b. Use a numerical evidence to make a conjecture about the limit of the sequence.

$a_1 = 1$

$a_2 = \frac{5}{8} \approx 0.62$

$a_3 = \frac{14}{27} \approx 0.51\ldots$

$a_4 = \frac{30}{64} \approx 0.46$

$a_5 = \frac{55}{125} \approx 0.44$

$a_6 = \frac{91}{216} \approx 0.42$

$\vdots$

Guess $\lim_{n\to\infty} a_n = 0.4$

c. Confirm your conjecture by expressing a_n in closed form and calculating the limit.

$$a_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{2}{6} = \frac{1}{3}$$

iii.Page 657: Q.40: Use numerical evidence to make a conjecture about the limit of the sequence $\left\{\left(\frac{1+n}{2n}\right)^n\right\}_{n=1}^{\infty}$, and then use the Squeezing theorem for Sequences to confirm that your conjecture is correct.

$$\lim_{n\to\infty}\left(\frac{1+n}{2n}\right)^n = \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{1+n}{n}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \cdot \left(1+\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= 0 \cdot e$$

$$= 0$$