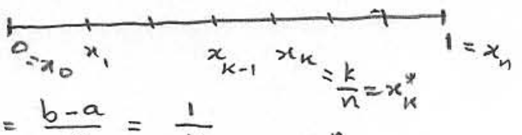
Solution MATH 102 [Homework 4]

1. Page 426: Q. 66. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2};[0,1]$, by interpreting it as Riemann sum in which the interval [0,1] is divided into n subinterval of equal width.

i-



$\Delta x = \frac{b-a}{n} = \frac{1}{n}$

ii- $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x$

$" \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}\cdot n\cdot\frac{1}{n}$

$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$

$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+(x_k^*)^2}\,\Delta x$

$\boxed{\text{*- Here: } f(x_k^*)=\frac{1}{1+(x_k^*)^2}\ ;\ \Delta x=\frac{1}{n}}$

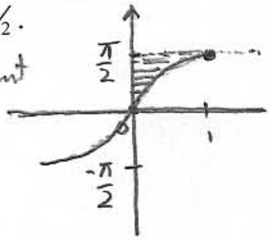
$= \int_0^1 \frac{1}{1+x^2}\,dx$

$= \tan^{-1}x\Big]_0^1 = \tan^{-1}1-\tan^{-1}0 = \frac{\pi}{4}-0$

$= \frac{\pi}{4}$

2. Page 444: Q. 28. Find the area of the region enclosed by the graphs of $y=\sin^{-1}x, x=0,\text{and } y=\pi/2$.

(Note: Graph is important to understand the problem)



i- $y=\sin^{-1}x \Rightarrow x=\sin y$

ii- Area of shaded region

$= \int_0^{\pi/2}\sin y\,dy = -\cos y\Big]_0^{\pi/2}$

$= -\left[\cos\frac{\pi}{2}-\cos 0\right]$

$= -[0-1]$

$= 1$

3. Evaluate

i. $\int_0^8 \frac{dx}{\sqrt{2x+9}(x+13)}$;

Put $\sqrt{2x+9}=u$
$\Rightarrow du = dx/\sqrt{2x+9}$
Also $x=\frac{u^2-9}{2}$
$x=0\Rightarrow u=3$
$x=8\Rightarrow u=5$

$= \int_3^5 \frac{du}{\frac{u^2-9}{2}+13} = 2\int_3^5\frac{du}{u^2+17}$

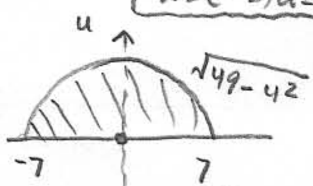
$= \frac{2}{\sqrt{17}}\tan^{-1}\frac{u}{\sqrt{17}}\Big]_3^5 = \frac{2}{\sqrt{17}}\left[\tan^{-1}\frac{5}{\sqrt{17}}-\tan^{-1}\frac{3}{\sqrt{17}}\right]$

ii. $\int_{e^{-7}}^{e^7}\frac{\sqrt{49-(\ln x)^2}\,dx}{x}$;

Put $u=\ln x$
$du=\frac{dx}{x}$
$x=e^{-7}\Rightarrow u=-7$
$x=e^7\Rightarrow u=7$

$= \int_{-7}^{7}\sqrt{49-u^2}\,du$

= Area of Semicircle



$= \frac{\pi}{2}(r^2)$

r = 7

$= \frac{\pi}{2}(7) = \frac{7\pi}{2}$

iii. $\int_0^1\frac{x^2dx}{\sqrt{9-5x}}$;

Put $u=9-5x$
$du=-5dx$
$x=0\Rightarrow u=9$
$x=1\Rightarrow u=4$
$x=\frac{u-9}{5}$

$= -\frac{1}{5}\int_9^4\frac{1}{\sqrt{u}}\left(\frac{u-9}{5}\right)^2du$

$= \frac{-1}{125}\int_9^4\frac{1}{\sqrt{u}}(u^2-18u+81)\,du = \frac{-1}{125}\int_9^4(u^{3/2}-18u^{1/2}+81u^{-1/2})\,du$

$= \frac{-1}{125}\left[\frac{2}{5}u^{5/2}-18\cdot\frac{2}{3}u^{3/2}+81\cdot 2u^{1/2}\right]_9^4$

$= \frac{-1}{125}\left[\frac{2}{5}(2^5-3^5)-12(2^3-3^3)+162(2-3)\right] = \ldots\ldots$

iv. $\int_1^{\sqrt{2}}\frac{w\,dw}{3+w^4}$;

Put $w^2=u$
$2w\,dw=du$
$w=1\Rightarrow u=1$
$w=\sqrt{2}\Rightarrow u=2$

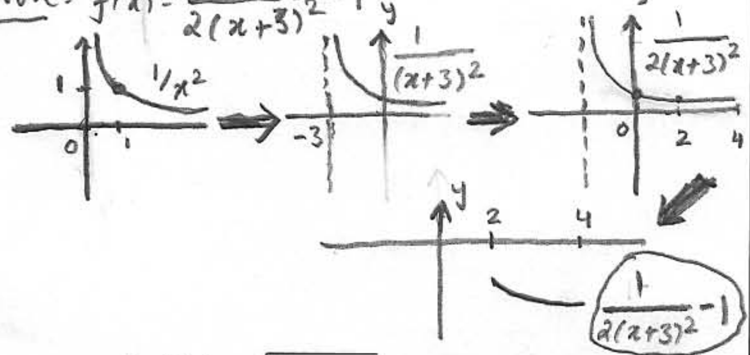
$= \frac{1}{2}\int_1^2\frac{du}{3+u^2}$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{u}{\sqrt{3}}\right]_1^2 = \frac{1}{2\sqrt{3}}\left[\tan^{-1}\frac{2}{\sqrt{3}}-\tan^{-1}\frac{1}{\sqrt{3}}\right]$

4. Sketch the region under the curve $y = f(x)$ over the interval $[a,b]$: (Use basic graphs with Horizontal & Vertical Shifts)

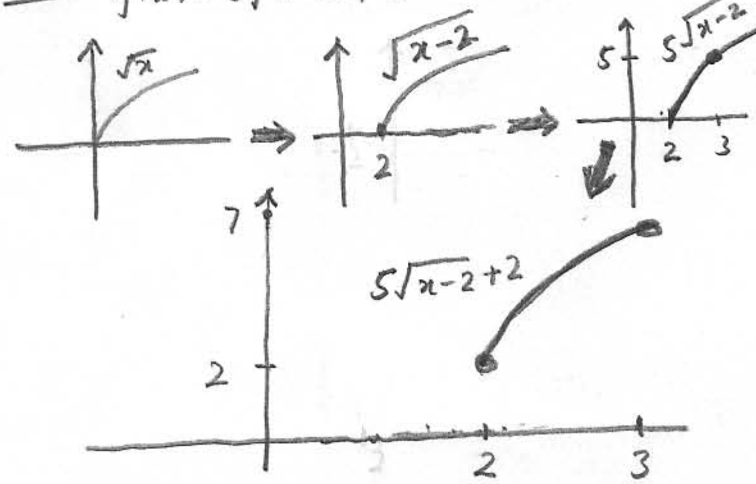
i. $f(x) = \dfrac{1}{(4x+12)^2} - 1, \qquad [a,b] = [2,4]$

Note: $f(x) = \dfrac{1}{2(x+3)^2} - 1$



ii. $f(x) = \sqrt{25x-50} + 2, \qquad [a,b] = [2,6]$

Note: $f(x) = 5\sqrt{x-2} + 2$



5. i. Page 455: Q 4(c): Given that $\ln a = 9$, find $\int_1^{2/a} \frac{1}{t} dt$.

$$\int_1^{2/a} \frac{1}{t} dt = \ln t \Big]_1^{2/a} = \ln\frac{2}{a} - \ln 1$$

$$= \ln 2 - \ln a - 0$$

$$= \boxed{\ln 2 - 9}$$

ii. 7(e): Simplify $\exp(3\ln x)$ and state values of x for which simplified expression is valid.

$\exp(3\ln x)$ $\qquad \boxed{x>0}$

$= e^{3\ln x} = e^{\ln x^3} = x^3$

Ans $\boxed{x^3 \; ; \quad x>0}$

iii. 7(h): Simplify $e^{x-\ln x}$ and state values of x for which simplified expression is valid.

$e^{x-\ln x}$ $\qquad \boxed{x>0}$

$= e^{x} - e^{-\ln x} = e^{x} - e^{\ln\frac{1}{x}}$

$= \boxed{e^{x} - \frac{1}{x}, \quad x>0}$

iv. 10(b): Express x^{2x} as power of e.

$$x^{2x} = e^{\ln x^{2x}} = e^{2x\ln x}$$

v. 22: Evaluate:

a. $\dfrac{d}{dx}\displaystyle\int_x^0 (t^2+1)^{40} dt$; b. $\dfrac{d}{dx}\displaystyle\int_{1/x}^{\pi} \cos^3 t\, dt$

$$(a)\ \frac{d}{dx}\int_0^x (t^2+1)^{40} dt = -\frac{d}{dx}\int_0^x (t^2+1)^{40} dt$$

$$= -(x^2+1)^{40}$$

$$(b)\ \frac{d}{dx}\int_{1/x}^{\pi} \cos^3 t\, dt = -\frac{d}{dx}\int_{\pi}^{1/x} \cos^3 t\, dt$$

$$= -\cos^3\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -\cos^3\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$

$$= \boxed{\frac{1}{x^2}\cos^3\left(\frac{1}{x}\right)}$$

vi. 25: Evaluate:

a. $\dfrac{d}{dx}\int_{x^2}^{x^3}\sin^2 t\,dt$; b. $\dfrac{d}{dx}\int_{-x}^{x}\dfrac{1}{1+t}dt$

(a) $\dfrac{d}{dx}\int_{x^2}^{x^3}\sin^2 t\,dt = \sin^2(x^3)\dfrac{d}{dx}(x^3) - \sin^2(x^2)\dfrac{d}{dx}(x^2)$

$= \boxed{3x^2\sin^2(x^3) - 2x\sin^2(x^2)}$

(b) $\dfrac{d}{dx}\int_{-x}^{x}\dfrac{1}{1+t}dt$

$= \ln|1+x|\dfrac{dx}{dx} - \ln|1-x|\dfrac{d(-x)}{dx}$

$= \ln|1+x| + \ln|1-x|$

$= \ln\left|\dfrac{1+x}{1-x}\right|$

vii. 30. Express

$$F(x)=\int_0^x f(t)dt, \text{ where } f(t)=\begin{cases} x & 0\le x\le 2 \\ 2, & x>2\end{cases}$$

in a piecewise form without integral sign.

Case (i) $\boxed{0 \le x \le 2}$

$F(x) = \int_0^x f(t)dt = \int_0^x t\,dt = \left.\dfrac{t^2}{2}\right]_0^x = \dfrac{x^2}{2}$

Case (ii) $\boxed{x > 2}$

$F(x) = \int_0^x f(t)dt = \int_0^2 f(t)dt + \int_2^x f(t)dt$

$= \int_0^2 x\,dt + \int_2^x 2\,dt$

$= \dfrac{4}{2} + 2t\Big]_2^x = 2 + 2(x-2)$

$= 2x - 2$

Ans
$$\boxed{F(x) = \begin{cases} \dfrac{x^2}{2} & 0 \le x \le 2 \\ 2x-2 & x > 2\end{cases}}$$

6. Use Theorem 6.9.6 to evaluate:

i. $\lim\limits_{x\to 0}(1+2x)^{3/5x}$; ii. $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{7x}\right)^{3x/8}$

(i) $\lim\limits_{x\to 0}(1+2x)^{\frac{3}{5x}}$

$= \lim\limits_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^{\frac{3}{5}(2)}$

$= \left[\lim\limits_{x\to 0}(1+2x)^{\frac{1}{2x}}\right]^{\frac{6}{5}}$

$= e^{6/5}$

(ii) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{7x}\right)^{3x/8}$

$= \lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{\frac{7}{2}x}\right)^{\frac{7x}{2}}\right]^{\frac{3}{8}\left(\frac{2}{7}\right)}$

$= \left[\lim\limits_{x\to\infty}\left(1+\dfrac{1}{\frac{7}{2}x}\right)^{\frac{7}{2}x}\right]^{\frac{3}{28}}$

$= e^{3/28}$