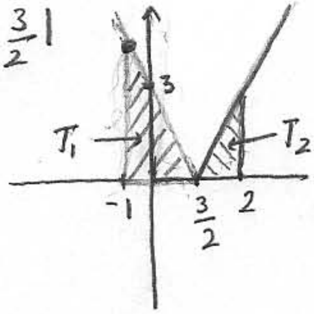
# MATH 102 [Solution Homework 3]

1. i. Q 13(c): Sketch the region whose signed area is represented by the definite integral $\int_{-1}^{2}|2x-3|dx$. Then evaluate the integral using appropriate formula from geometry.

i- $|2x-3| = 2|x-\frac{3}{2}|$




ii- $\int_{-1}^{2}|2x-3|dx$

$= \text{Area of } \Delta T_1 + \text{Area of } \Delta T_2$

$= \frac{1}{2}\left(\frac{5}{2}\right)(5) + \frac{1}{2}\left(\frac{1}{2}\right)(1)$

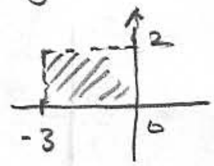
$= \frac{25}{4} + \frac{1}{4} = \frac{26}{4} = \frac{13}{2}$

While going through the solution, check the calculations carefully and fix the mistake of any.

ii. Q22 (a): Use the properties of Definite integral and appropriate formula from geometry to evaluate $\int_{-3}^{0}\left(2-\sqrt{9-x^2}\right)dx$.

(i) $\int_{-3}^{0}(2-\sqrt{9-x^2})dx = \int_{-3}^{0}2\,dx - \int_{-3}^{0}\sqrt{9-x^2}\,dx$

(ii) $\int_{-3}^{0}2\,dx = 2(3) = 6$



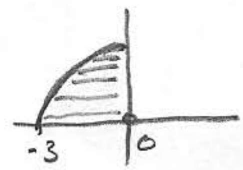

(iii) $\int_{-3}^{0}\sqrt{9-x^2}\,dx$

$= \frac{1}{4}\pi(3)^2$

$= \frac{9\pi}{4}$

$y = \sqrt{9-x^2}$

$y^2 + x^2 = 9$

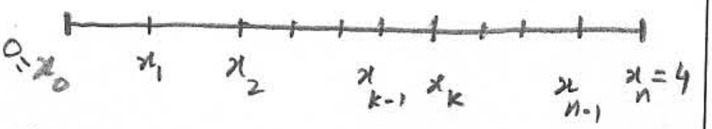



(iv) Ans

$\int_{-3}^{0}(2-\sqrt{9-x^2})dx = 6 - \frac{9\pi}{4}$

iii. Q.33: Evaluate $\int_0^4 \sqrt{x}dx$ by using the definition of Riemann Integral: $\int_0^4 f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x^*)\Delta x_k$. Here, the subintervals of unequal length are given by the following partition of [0,4] : $0 < \frac{4(1)^2}{n^2} < \frac{4(2)^2}{n^2} < \ldots < \frac{4(n-1)^2}{n^2} < 4$; and $x_k^*$ is the right end point of the $k^{th}$ subinterval.

i.




Here
$x_0 = 0$
$x_1 = 4(1)^2/n^2$
$x_2 = 4(2)^2/n^2$
$\vdots$
$x_{k-1} = 4(k-1)^2/n^2$
$x_k = 4k^2/n^2$
$\vdots$
$x_{n-1} = 4(n-1)^2/n^2$
$x_n = 4$

(ii) Considere $[x_{k-1}, x_k]$ Then

(a) $\Delta x_k = x_k - x_{k-1} = \frac{4k^2}{n^2} - \frac{4(k-1)^2}{n^2}$

$= \frac{4}{n^2}[k^2 - k^2 + 2k - 1] = \frac{4}{n^2}(2k-1)$

(b) $x_k^* =$ Rightend pt. of $[x_{k-1}, x_k]$

$= x_k = 4k^2/n^2$

(c) $f(x_k^*) = \sqrt{x_k^*} = \sqrt{4k^2/n^2} = 2k/n$

(iii) $\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} \frac{2k}{n} \cdot \frac{4}{n^2}(2k-1)$

$= \frac{8}{n^3} \sum_{k=1}^{n} (2k^2 - k)$

$= \frac{8}{n^3}\left[\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$= \frac{8n(n+1)}{n^3}\left[\frac{2n+1}{3} - \frac{1}{2}\right]$

$= 8n(n+1)[4n-1]/6n^3$

(iv) $\int_0^4 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

$= \lim_{n\to\infty} \frac{8n(n+1)(4n-1)}{6n^3} = \frac{8(4)}{6}$

$= \frac{16}{3}$

2. Sketch the region enclosed between the graphs of $f(x) = |x|$ and $g(x) = 1 + \sqrt{1 - x^2}$. Using geometrical argument find area of the region.

i- $f(x) = |x|$

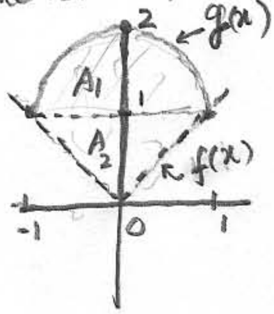
ii- $g(x) = 1 + \sqrt{1 - x^2}$

$\Rightarrow y = 1 + \sqrt{1 - x^2}$

$(y-1)^2 = 1 - x^2$

$x^2 + (y-1)^2 = 1$

$\Rightarrow g(x) = 1 + \sqrt{1 - x^2}$ is upper semi circle with centre at $(0,1)$ and radius 1.



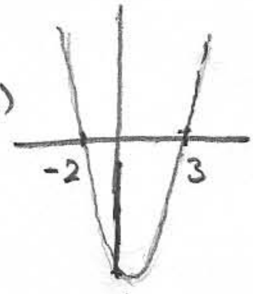

iii- Area $= A_1 + A_2$

$= \frac{\pi}{2}(1)^2 + \frac{1}{2}(2)(1)$

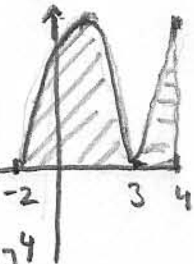
$= \frac{\pi}{2} + 1$

3. Sketch the graph of $y = |x^2 - x - 6|$ and find the area under this curve above the interval $[-2, 4]$.

i- Graph of $y = x^2 - x - 6$
$= (x-3)(x+2)$




ii- Graph of $y = |x^2 - x - 6|$




iii- Area $= \int_{-2}^{4} |x^2 - x - 6|\,dx$

$= \int_{-2}^{3} -(x^2 - x - 6)\,dx + \int_{3}^{4} (x^2 - x - 6)\,dx$

$= -\left[\frac{x^3}{3} - \frac{x^2}{2} - 6x\right]_{-2}^{3} + \left[\frac{x^3}{3} - \frac{x^2}{2} - 6x\right]_{3}^{4}$

$= \frac{59}{3}$ (Please check this number)

4. Evaluate the integrals:

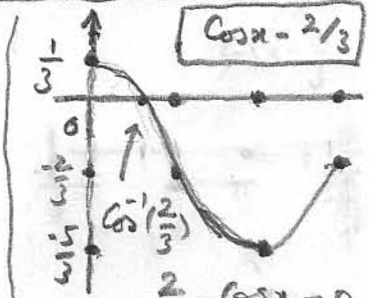
i. $\displaystyle\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dt}{t\sqrt{t^2-1}}$

$= \left[\sec^{-1} t\right]_{-\sqrt{2}}^{-2/\sqrt{3}}$

$= \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) - \sec^{-1}(-\sqrt{2})$

$= \frac{5\pi}{6} - \frac{3\pi}{4} = \frac{\pi}{12}$

ii. $\displaystyle\int_0^{2\pi/3} \left|\frac{2}{3} - \cos x\right| dx$




$\frac{2}{3} - \cos x = 0$

$\cos x = \frac{2}{3}$

$x = \cos^{-1}\left(\frac{2}{3}\right)$

$= \displaystyle\int_0^{\cos^{-1}\frac{2}{3}} \left(\cos x - \frac{2}{3}\right) dx + \int_{\cos^{-1}\frac{2}{3}}^{2\pi/3} -\left(\cos x - \frac{2}{3}\right) dx$

$= \left[\sin x - \frac{2}{3}x\right]_0^{\cos^{-1}(2/3)} - \left[\sin x - \frac{2}{3}x\right]_{\cos^{-1}\left(\frac{2}{3}\right)}^{2\pi/3}$

$= 2\left[\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right) - \frac{2}{3}\cos^{-1}\left(\frac{2}{3}\right)\right] - \sin\frac{2\pi}{3} - \frac{4\pi}{9}$

$= 2\left[\frac{\sqrt{5}}{3} - \frac{2}{3}\cos^{-1}\left(\frac{2}{3}\right)\right] + 1 - \frac{4\pi}{9}$

iii. $\displaystyle\int_{-1}^{2} f(x)dx$ where $f(x) = \begin{cases} \sqrt{1-x}, & -1 \le x \le 0 \\ 1-x^2, & 0 \le x \le 2 \end{cases}$

$\displaystyle\int_{-1}^{2} f(x)dx = \int_{-1}^{0} \sqrt{1-x}\,dx + \int_0^2 (1-x^2)\,dx$

$= \left[\frac{2}{3}(1-x)^{3/2}\right]_{-1}^{0} + \left[x - \frac{x^3}{3}\right]_0^2$

$= \frac{2}{3}\left(1 - 2^{3/2}\right) + \left(2 - \frac{8}{3}\right)$

$= \frac{2}{3} + \frac{2}{3} - \frac{2}{3}\cdot 2\sqrt{2}$

$= \frac{2}{3}\left[2 - 2\sqrt{2}\right]$

5. Let $F(x) = \displaystyle\int_{-1}^{x} \sqrt{\sin^2(t^2 - \pi/4) + 3}\,dt$. Find $F(-1)$, $F'\left(\sqrt{\frac{\pi}{2}}\right)$, $F''\left(\sqrt{\frac{\pi}{2}}\right)$.

i- $F(-1) = 0$

ii- $F'(x) = \sqrt{\sin^2\left(t^2 - \frac{\pi}{4}\right) + 3}$

$\Rightarrow F'\left(\sqrt{\frac{\pi}{2}}\right) = \sqrt{\sin^2\left(\frac{\pi}{4}\right) + 3}$

$= \sqrt{\frac{1}{2} + 3} = \sqrt{5}/\sqrt{2}$

iii- $F''(x) = \dfrac{\sin(t^2 - \pi/4)\cos\left(t^2 - \frac{\pi}{4}\right)(2t)}{\sqrt{\sin^2\left(t^2 - \frac{\pi}{4}\right) + 3}}$

$F''\left(\sqrt{\frac{\pi}{2}}\right) = \dfrac{\sin(\pi/4)\cos(\pi/4)\cdot 2\sqrt{\pi}/\sqrt{2}}{\sqrt{5}/\sqrt{2}}$

$= \frac{1}{2}\cdot\frac{2\sqrt{\pi}\cdot}{\sqrt{5}} = \sqrt{\frac{2\pi}{5}}$

6. Find the values of $x^*$ in the interval $[1,3]$ that satisfies the conclusion of the Mean Value Theorem of Integrals for the function $f(x) = x^3$.

M.V.T. $\displaystyle\int_1^3 x^3 dx = (x^*)^3 (3-1)$

$\left.\frac{x^4}{4}\right]_1^3 = (x^*)^3 \cdot 2$

$\frac{1}{4}(81-1) = 2(x^*)^3$

$(x^*)^3 = \frac{1}{8}(80)$

$(x^*)^3 = 10$

$\boxed{x^* = \sqrt[3]{10}}$