# 10.3 Monotone Sequences
**[Read Examples 1 to 6 p.658-662]**

**1. *New Concept*: "Increasing Sequence" $\{a_n\}_{n=1}^{\infty}$:**

$a_1 \le a_2 \le a_3 \le \cdots;$ *i.e.,* $a_{n+1} \ge a_n$

**Examples:** $\{2^k\}_{k=1}^{\infty}, \{2n-5\}_{n=1}^{\infty}, \{\frac{m}{m+1}\}_{m=1}^{\infty}$

**2. *New Concept*: Eventually Increasing Sequence**
A sequence which is increasing after a Finite Number of Terms

**Examples:** $\{\frac{n!}{3^n}\}_{k=1}^{\infty}, \{\frac{n^2}{n^2-2}\}_{n=1}^{\infty}, \{\frac{m}{m+1}\}_{m=1}^{\infty}$

**3. Tests for Increasing Sequence:**
**i. Test # 1:** $a_{n+1} - a_n \ge 0$.
**ii. Test # 2:** $\frac{a_{n+1}}{a_n} \ge 1$.
**iii. Test # 3:** $f'(x) \ge 0 \Rightarrow f(n) = a_n$ is $\uparrow$.

**Ex: Check if the following sequences are increasing**

i. $\{\frac{n+1}{2n+3}\}_{n=1}^{\infty}$, Use Test # 1

ii. $\{\frac{n!}{3^n}\}_{n=1}^{\infty}$, Use Test # 2

iii. $\{\tan^{-1} n\}_{n=1}^{\infty}$; Use Test # 3

**i. Bracket form:** $\{a_n\}_{n=1}^{\infty}$.
**ii. Expanded form:** $a_1, a_2, a_3, \ldots$

**4. *New Concept*: "Decreasing Sequence" $\{a_n\}_{n=1}^{\infty}$:**

$a_1 \ge a_2 \ge a_3 \ge \cdots;$ *i.e.,* $a_n \ge a_{n+1}$

- **Express (2) & (3) for "Decreasing Sequence".**

**Ex: Check if the following sequences are decreasing**

i. $\{3 - \frac{1}{n}\}_{n=1}^{\infty}$, ii. $\{\frac{1+2^n}{2^n}\}_{n=1}^{\infty}$, iii. $\{e^{-n}n^2\}_{n=1}^{\infty}$

**5. *Another Name for Increasing or Decreasing Sequence*: "Monotone Sequence"**

**7. Concepts:**
- **Upper Bound of a Sequence**
- **Lower Bound of a Sequence.**
- **Sequences Bounded Above.**
- **Sequences Bounded Below.**

**8. Tests for Convergence of Monotone Sequences**

**Part i.** Given: $\{a_n\}_{n=1}^{\infty}$ is an eventually increasing sequence. Then

**(a)** $\{a_n\}_{n=1}^{\infty}$ **converges** to a number $L \le M$ if it has an Upper Bound $M$.

**(b)** $\lim_{n\to\infty} a_n = \infty$ if it is not bounded above .

**Part ii.** Given: $\{a_n\}_{n=1}^{\infty}$ is an eventually decreasing sequence. Then

**(a)** $\{a_n\}_{n=1}^{\infty}$ **converges** to a number $L \ge M$ if it has a Lower Bound $M$.

**(b)** $\lim_{n\to\infty} a_n = -\infty$. if it is not bounded below.

**9. Exercises (*Important Limits*):**

i. Show that $\{\frac{4^n}{n!}\}_{n=1}^{\infty}$ converges [Use part (ii)] . Also, find its limit. [Read Example 6, page: 662].

ii. Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ [Use l'Hopital Rule]

ii. Evaluate $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}}$ [Use (i)]

iii. Find $\lim_{n\to\infty} \sum_{k=1}^{n} [\frac{1}{2k+1} - \frac{1}{2k+5}]$.

**10.** Does the limit of following sequences exist. If so, evaluate:

i. $\{\frac{2}{n+3}\}_{n=1}^{\infty}$. ii. $\left\{\frac{(-1)^n}{n+3}\right\}_{n=1}^{\infty}$ iii. $\left\{\frac{(-\frac{1}{2})^n}{n+3}\right\}_{n=1}^{\infty}$ iv. $\left\{\frac{(-2)^n}{n+3}\right\}_{n=1}^{\infty}$

v. $\{\frac{\cos n}{n+3}\}_{n=1}^{\infty}$ vi. $\{\frac{\tan^{-1} n}{2n+5}\}_{n=1}^{\infty}$ vii. $\{\frac{\tan n}{2n+5}\}_{n=1}^{\infty}$

viii. $\{n \tan \frac{1}{8n}\}_{n=1}^{\infty}$ ix. $\left\{\sum_{k=1}^{n} (-1)^k\right\}_{n=1}^{\infty}$

**11.** i. Comparing appropriate areas, show that

$$\int_1^n \ln x dx < \ln(n!) < \int_1^{n+1} \ln x dx$$

ii. Show that

$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$, $n > 1$. [Use (i)]

iii. Show that

$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$. [Use (ii)]

iv. Show that

$\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. [Use (ii)]