King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Semester(Term 041)

MATH 101

Second Major Exam

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1. Use the limit definition of the derivative to compute the derivative of $f(x) = \dfrac{2}{\sqrt{x}}$.

(8 points)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(1) $\boxed{[1]\ f(x+h) = \dfrac{2}{\sqrt{x+h}}}$ $\boxed{[2]\ f(x) = \dfrac{2}{\sqrt{x}}}$ [3] $f(x+h) - f(x) = \dfrac{2}{\sqrt{x+h}} - \dfrac{2}{\sqrt{x}}$

$$\overset{(1)}{=} \frac{2\sqrt{x} - 2\sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}} = \frac{2(\sqrt{x} - \sqrt{x+h})}{\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \quad (1)$$

$$= \frac{2(x - (x+h))}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \boxed{\frac{-2h}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}} \quad (1)$$

$$f'(x) = \lim_{h \to 0} \overset{(1)}{} \frac{-2\cancel{h}}{\cancel{h}\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-2}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \quad (1)$$

$$= \frac{-2}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} \quad (1)$$

$$= \frac{-\cancel{2}}{x(\cancel{2}\sqrt{x})}$$

$$= \boxed{\frac{-1}{x\sqrt{x}}} = \boxed{\frac{-1}{x^{3/2}}} \quad (1)$$

2. Assume that

$$f(x) = \begin{cases} \dfrac{1}{4}x^3 - \dfrac{1}{2}x^2, & \text{if } \quad x \geq 2 \\[2ex] \dfrac{-6x-6}{x^2+2}, & \text{if } \quad x < 2 \end{cases}$$

Determine if f is differentiable at $x = 2$, i.e. determine $f'(2)$ if it exists.

(10 points)

Method 1

First, determine if f is continuous at $x = 2$, i.e. if $\boxed{f(2) = \lim_{x \to 2} f(x).}$

$\boxed{1}$ $f(2) = \frac{1}{4}(2)^3 - \frac{1}{2}(2)^2 = \frac{8}{4} - \frac{4}{2} = 2 - 2 = 0$ (1)

$\boxed{2}$ $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{4}x^3 - \frac{1}{2}x^2 = \frac{1}{4}(2)^3 - \frac{1}{2}(2)^2 = 0$ (2)

$\boxed{3}$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{-6x-6}{x^2+2} = \frac{-6(2)-6}{(2)^2+2} = \frac{-18}{6} = -3$ (2)

$\boxed{4}$ $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x) \quad \therefore \quad \lim_{x \to 2} f(x)$ DNE (2)

and hence f is not continuous (1) at $x = 2$. Therefore,

f is not differentiable at $x = 2$. (2)

Method 2

$\boxed{1}$ $f'_+(2) = \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$ (1) $= \lim_{h \to 0^+} \frac{\frac{1}{4}(2+h)^3 - \frac{1}{2}(2+h)^2 - 0}{h}$ (1)

$= \lim_{h \to 0^+} \frac{\frac{1}{2}(2+h)^2\left[\frac{1}{2}(2+h) - 1\right]}{h}$ (1) $= \lim_{h \to 0^+} \frac{\frac{1}{2}(2+h)^2\left[\cancel{1} + \frac{1}{2}h \cancel{-1}\right]}{h} = \lim_{h \to 0^+} \frac{\frac{1}{2}(2+h)^2 \cdot \frac{1}{2}\cancel{h}}{\cancel{h}}$ (1)

$= \lim_{h \to 0^+} \frac{(2+h)^2}{4} = 1$ (1)

$\boxed{2}$ $f'_-(2) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$ (1) $= \lim_{h \to 0^-} \frac{\frac{-6(2+h)-6}{(2+h)^2+2} - 0}{h}$ (1) $= \lim_{h \to 0^-} \frac{-6[2+h+1]}{h[(2+h)^2+2]}$ (1)

$= \lim_{h \to 0^-} \frac{-6[3+h]}{h[4+4h+h^2+2]} = \lim_{h \to 0^-} \frac{-6(3+h)}{h(h^2+4h+6)}$ (1) $= \frac{-6(3+0)}{0(0+0+6)} = \frac{-18}{0}$ DNE (1)

$\therefore$ f is not differentiable at $x = 2$

see next page for an important Remark!

Remark what follows is a common INCORRECT attempt to solve this problem using another method.

For $x>2$ $\quad f'(x)=\frac{3}{4}x^2-x$

$$\therefore \lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+}\left(\frac{3}{4}x^2-x\right) = \frac{3}{4}(2)^2-(2) = 3-2=\boxed{1}$$

For $x<2$ $\quad f'(x)=\frac{(-6)(x^2+2)-(2x)(-6x-6)}{(x^2+2)^2}$

$$= \frac{-6x^2-12+12x^2+12x}{(x^2+2)^2}$$

$$= \frac{6x^2+12x-12}{(x^2+2)^2}$$

$$\therefore \lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} \frac{6x^2+12x-12}{(x^2+2)^2} = \frac{6(4)+12(2)-12}{(4+2)^2} = \frac{36}{36} = 1$$

$$\therefore \lim_{x\to 2^+} f'(x) = \lim_{x\to 2^-} f'(x) = \lim_{x\to 2} f'(x) = 1$$

nothing wrong yet

An INCORRECT Conclusion would be that

$f'(2)=1$. If f' were continuous at $x=2$, this would be a valid method to compute $f'(2)$.

3. For each of the following functions, compute the derivative. Show all work including each step in your derivation.

(5 points each)

(a) $y = \left(\frac{x}{4} + x^{-5}\right)^{\frac{1}{2}}$.

$y = \sqrt{\frac{x}{4} + x^{-5}}$ (1)

$y' = \frac{1}{2\sqrt{\frac{x}{4} + x^{-5}}} \cdot \left(\frac{1}{4} - 5x^{-6}\right)$ (2)

(2)

(b) $y = \left(\frac{8x - x^6}{x^3}\right)^{-\frac{4}{5}}$.

(1) (1) (3)

$y' = \underbrace{\frac{-4}{5}}\underbrace{\left(\frac{8x - x^6}{x^3}\right)^{-\frac{9}{5}}} \cdot \frac{(8 - 6x^5)x^3 - 3x^2(8x - x^6)}{x^6}$

$= \frac{-4}{5}\left(\frac{8x - x^6}{x^3}\right)^{-\frac{9}{5}} \cdot \frac{8x^3 - 6x^8 - 24x^3 + 3x^8}{x^6}$

$= \frac{-4}{5}\left(\frac{8x - x^6}{x^3}\right) \cdot \frac{-16x^3 - 3x^8}{x6}$

(c) $y = \sec^2(x^4)\tan^3(x^4)$. (2)

$y' = \underbrace{\left[2\sec(x^4) \cdot \sec(x^4)\tan(x^4) \cdot 4x^3\right]}\tan^3 x^4 +$ ← right product rule

$\underbrace{\left[3\tan^2(x^4) \cdot \sec^2(x^4) \cdot 4x^3\right]}\sec^2(x^4)$

(2) (1)

(d) $y = \sqrt{\sin(7x + \cos 5x)}$.

$$y' = \underset{1}{\frac{1}{2\sqrt{\sin(7x+\cos 5x)}}} \cdot \underset{2}{\cos(7x+\cos 5x)} \cdot \underset{2}{(7 - 5\sin 5x)}$$

(e) $y = \tan^3 \sqrt{\cot 7x}$.

$$y' = \underset{1}{3\tan^2\sqrt{\cot 7x}} \cdot \underset{1}{\sec^2\sqrt{\cot 7x}} \cdot \underset{1}{\frac{1}{2\sqrt{\cot 7x}}} \cdot \underset{1}{-\csc^2 7x} \cdot \underset{1}{7}$$

(f) $y = \dfrac{1}{x\sqrt{x^2+1}}$.

$$y = \left(x\sqrt{x^2+1}\right)^{-1}$$

$$y' = -\underset{1}{\left(x\sqrt{x^2+1}\right)^{-2}} \cdot \left(\underset{1}{\sqrt{x^2+1}} + \underset{1}{\frac{1}{2\sqrt{x^2+1}}} \cdot \underset{1}{2x} \cdot \underset{1}{x}\right)$$

4. The point $(1,2)$ lies on the hyperbola with equation $x^2+2xy-y^2+x=2$. Using implicit differentiation, determine the equation of the tangent line to the hyperbola at $(1,2)$.

(9 points)

$\boxed{1}$ The standard eq of the tangent line to the hyperbola at $(1,2)$ is

$$y-2 = f'(x_0, y_0)(x-1) \quad 1$$

$\boxed{2}$ Using implicit differentiation, we have

$$2x+2y+2xy'-2yy'+1=0 \quad 2$$

$$2xy'-2yy' = -1-2x-2y$$

$$(2x-2y)y' = -1-2x-2y$$

$$\therefore\ y' = \frac{-1-2x-2y}{2x-2y} \quad 2$$

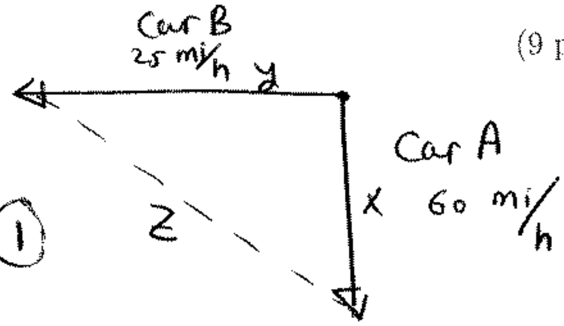
$$y'\Big|_{(1,2)} = f'(1,2) = \frac{-1-2(1)-2(2)}{2(1)-2(2)} = \frac{-1-2-4}{2-4}$$

$$= \frac{-7}{-2} = \boxed{\frac{7}{2}} \quad 2$$

$\therefore$ The eq of the tangent line to $x^2+2xy-y^2+x=2$ at $(1,2)$ is $\boxed{y-2 = \frac{7}{2}(x-1)}$ 2

5. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

(9 points)



2

$\boxed{1}$ Given $\frac{dx}{dt} = 60$ mi/h, $\frac{dy}{dt} = 25$ mi/h 1

$\boxed{2}$ asked to find $\frac{dz}{dt}$ after 2 hours.

$\boxed{3}$ By Pythagorean Theorem

$$z^2 = x^2 + y^2 \quad 1$$

$$\therefore \cancel{2}z \frac{dz}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{1}{z}\left(x \frac{dx}{dt} + y \frac{dy}{dt}\right) \quad 1$$

If $t=2$, 1 $x = 120$ mi, 1 $y = 50$ mi

$\therefore z^2 = (120)^2 + (50)^2 = 6900$

1 $\therefore z = 130$

$$\therefore \frac{dz}{dt} = \frac{1}{130}(120 \cdot 60 + 50 \cdot 25) = 65 \text{ mi/h} \quad 1$$

6. Estimate $\sqrt{3.9}$. Show all work. (**Hint:** Use local linear approximation)

(9 points)

$\boxed{1}$ Let $f(x) = \sqrt{x}$ 1

$\boxed{2}$ Local linear approximation:

1 $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ for x close (near) to x_0

$\boxed{3}$ $x_0 = 4$ 1

$f(x_0) = f(4) = \sqrt{4} = 2$ 1

1 $f'(x) = \dfrac{1}{2\sqrt{x}}$ $\therefore f'(\underset{\downarrow}{4}) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$ 1

x_0

$\therefore \sqrt{x} \approx 2 + \frac{1}{4}(x-4)$ for x close to 4 2

$\therefore \sqrt{3.9} \approx 2 + \frac{1}{4}(3.9-4) = 2 + \frac{1}{4}(-.1) = 2 - .025$ 1

$= 1.975$

7. Let $f(x) = 4x - \sin 3x, \qquad -\frac{\pi}{6} < x < \frac{\pi}{6}.$

(9 points)

(a) Show that $f(x)$ is a one-to-one function.

(b) Show that $f^{-1}(x)$ is differentiable on the interval $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

(c) Find a formula for the derivative of f^{-1}.

(a) $f'(x) = 4 - 3\cos 3x$

$-1 \le \cos 3x \le 1$

$+3 \ge -3\cos 3x \ge -3$

$\therefore -3 \le -3\cos 3x \le 3$

$4-3 \le 4 - 3\cos 3x \le 4+3$

$1 \le \underbrace{4-3\cos 3x}_{\downarrow} \le 7$

$1 \le f'(x) \le 7$

$\therefore$ $f(x)$ is an increasing fun $\Rightarrow$ $\boxed{f \text{ is } 1-1}$

(b) (1) $D_f = \left(\frac{-\pi}{6}, \frac{\pi}{6}\right)$ open interval

(2) $f(x) = 4x - \sin 3x$ Diff on $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

poly everywhere

diff everywhere $\therefore$

(3) from (a) f 1-1

$\therefore$ $(f^{-1})'$ exist, $\forall$ (for all) $f'(x) \ne 0$ $(x \in D_f)$

but $f'(x) > 0$ in its domain.

Hence $(f^{-1})'$ is differentiable on $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

(c) $(f^{-1})' = \dfrac{1}{f'(f^{-1}(x))}$ let $y = f^{-1}(x) \Rightarrow x = f(y) = 4y - \sin 3y$

You can find y' by implicit diff too.

$f'(x) = 4 - 3\cos 3x$

$f'(f^{-1}(x)) = f'(y) = 4 - 3\cos 3y$

$\therefore$ $\boxed{(f^{-1})' = \dfrac{1}{4 - 3\cos 3y}}$

8. Solve for x, $\log_{e^2}^{(7-x)^2} + \ln(3x+5) = \ln(24x)$.

(8 points)

1 $\frac{1}{2}\log_e (7-x)^2 + \ln(3x+5) = \ln(24x)$

1 $\log_e (7-x) + \ln(3x+5) = \ln(24x)$

$\ln(7-x) + \ln(3x+5) = \ln(24x)$

1 $\ln(7-x)(3x+5) = \ln(24x)$

1 $(7-x)(3x+5) = 24x$

$21x+35-3x^2-5x = 24x$

$\therefore -3x^2+16x+35-24x=0$

$-3x^2-8x+35=0$

$3x^2+8x-35=0$

$(3x-7)(x+5)=0$

1 $\therefore x = \underbrace{\frac{7}{3}}$ or 1 $x=-5$ rejected 1 not in the domain of $\ln 24x$

$\therefore$ S-S $= \{\frac{7}{3}\}$ 1 (check it!)

9. Find the inverse of $f(x) = 3 - e^{x-2}$. $D_f = (-\infty, \infty)$ exponential fun

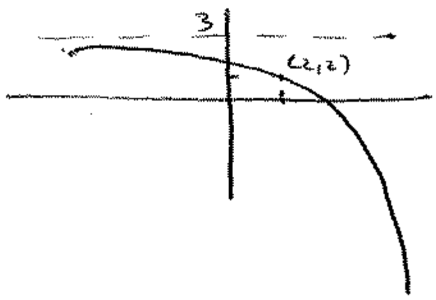
(8 points)

$\boxed{1}$ $y = 3 - e^{x-2}$ 1 1 $R_f = (-\infty, 3)$ from its graph

$\boxed{2}$ $y - 3 = -e^{x-2}$

$3 - y = e^{x-2}$

$\boxed{3}$ $\ln(3-y) = \ln e^{x-2}$ 1



$x - 2 = \ln(3-y)$

$x = 2 + \ln(3-y)$ 1

$\boxed{4}$ $y = 2 + \ln(3-x)$

$\boxed{5}$ $f^{-1}(x) = 2 + \ln(3-x)$, 2 $x < 3$ 2