King Fahd University of Petroleum and Minerals

Department of Mathematical Sciences

MATH 260 (Term 032)

Exam – II

(Dr. M. Samman)

Time: 90 Minutes

Solution

Name: ______________________________

Section: ______________________________

Serial: ______________________________

ID: ______________________________

1. Define each one of the following:

(a) Vector space

A vector space is a set V with two operations $+, \cdot$ such that

(a) The operation "+" is closed in V; $u+v \in V \quad \forall u,v \in V$.

1. $u+v=v+u \quad \forall u,v \in V$
2. $u+(v+w)=(u+v)+w \quad \forall u,v,w \in V$
3. $\exists$ an element $0 \in V$ such that $u+0=0+u=u$ for any $u \in V$
4. For each $u \in V$ $\exists$ $-u \in V$ such that $u+(-u)=-u+u=0$

(b) The "$\cdot$" is closed under scalar multiplication; $c.v \in V$ for any $v \in V, c \in \mathbb{R}$

5. $c.(u+v)=cu+cv \quad \forall u,v \in V, c \in \mathbb{R}$
6. $(c+d).u = c.u+du \quad \forall u \in V,\ c,d \in \mathbb{R}$
7. $c.(d.u) = (cd).u \quad \forall u \in V,\ c,d \in \mathbb{R}$
8. $1.u=u \quad \forall u \in V$.

(b) Subspace

A nonempty subset W of a vector space V is a subspace of V if W itself is a vector space.

(c) Dimension of a vector space.

The dimension of a vector space is the number of vectors in a basis for the vector space.

2. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & -2 \\ 1 & 4 & -2 & 4 \end{bmatrix}$

(a) Find $2A^T + I_4$.

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ 1 & -1 & 1 & -2 \\ 0 & 1 & -2 & 4 \end{bmatrix}$$

$$2A^T + I_4 = \begin{bmatrix} 2 & 0 & 2 & 2 \\ 4 & 2 & 6 & 8 \\ 2 & -2 & 2 & -4 \\ 0 & 2 & -4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 4 & 3 & 6 & 8 \\ 2 & -2 & 3 & -4 \\ 0 & 2 & -4 & 9 \end{bmatrix}$$

(b) Find A^{-1}, if it exists.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & 4 & -2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R_1+R_4]{(-1)R_1+R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_2+R_4]{-R_2+R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_2]{R_3+R_4}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & -3 & 1 & 1 \end{array}\right] \xrightarrow[-R_4]{2R_2+R_1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 4 & 3 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & -1 & -1 \end{array}\right] \xrightarrow[3R_4+R_3]{-R_3+R_1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 7 & 4 & 1 & -3 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 & -2 & -3 \\ 0 & 0 & 0 & 1 & 2 & 3 & -1 & -1 \end{array}\right] \xrightarrow[2R_4+R_2]{-7R_4+R_1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -10 & -20 & 4 & 7 \\ 0 & 1 & 0 & 0 & 3 & 6 & -1 & -2 \\ 0 & 0 & 1 & 0 & 5 & 8 & -2 & -3 \\ 0 & 0 & 0 & 1 & 2 & 3 & -1 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -10 & -20 & 4 & 7 \\ 3 & 6 & -1 & -2 \\ 5 & 8 & -2 & -3 \\ 2 & 3 & -1 & -1 \end{bmatrix}$$

3. (a) Are the vectors $u = (1,2,3)$, $v = (3,2,1)$, $w = (1,-1,6)$, linearly independent?

Note that $u, v, w \in \mathbb{R}^3$. So we compute the following determinant:

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 3 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 13 - 3(15) + 1(-4)$$

$$= -36$$

$$\neq 0$$

Hence the vectors u, v, w are linearly independent.

(b) Write the vector w as a linear combination of the standard basis.

$$w = (1,-1,6) = (1,0,0) + (0,-1,0) + (0,0,6)$$
$$= (1,0,0) - (0,1,0) + 6(0,0,1)$$

$$\therefore\ w = e_1 - e_2 + 6e_3$$

4. (a) Let $W = \{(x_1, x_2, x_3, x_4) : x_4 \geq 0\}$. Prove or disprove that W is a subspace of the vector space R^4.

Let $u, v \in W$, so

$$u = (x_1, x_2, x_3, x_4) \text{ , with } x_4 \geqslant 0$$
$$v = (y_1, y_2, y_3, y_4) \text{ , with } y_4 \geqslant 0 \qquad \text{Then}$$

$u+v = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4) \in W$. Since $x_4+y_4 \geqslant 0$.

Now, if c is a negative scalar, say $c=-1$, then $c.u \notin W$

for example take $u = (1,2,3,4) \in V$, $c=-1$, then

$$cu = -1(1,2,3,4) = (-1,-2,-3,-4) \notin W \text{ since } -4 \ngeq 0$$

Hence W is not a subspace of R^4.

(b) Let $v_1 = (1, 2, 1)$, $v_2 = (1, 0, 2)$, $v_3 = (1, 1, 0)$. Does the set $\{v_1, v_2, v_3\}$ form a basis for R^3?

Since v_1, v_2, v_3 are 3 vectors in R^3, all what we need is to check whether they are linearly independent or not. For this, we compute

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 3 \neq 0$$

Hence v_1, v_2, v_3 form a basis for R^3.

5. (a) If $S = \{u_1, u_2, \dots, u_k\}$ is a linearly independent set of vectors. Show that any vector $w \in Span\, S$ can be expressed in a unique way as a linear combination of the vectors $u_1, u_2, \dots, u_k$.

Let $w \in Span\, S$. Then

$$w = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k$$

Suppose also, $w = b_1 u_1 + b_2 u_2 + \cdots + b_k u_k$. Then

$$0 = (a_1 - b_1) u_1 + (a_2 - b_2) u_2 + \cdots + (a_k - b_k) u_k$$

But $u_1, u_2, \dots, u_k$ are linearly independent, thus

$a_1 - b_1 = 0$, $a_2 - b_2 = 0$, $\dots$, $a_k - b_k = 0$,

i.e. $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_k = b_k$,

which means that w has a unique representation as a linear comb. of $u_1, u_2, \dots, u_k$.

(b) Let A be $n \times n$ matrix. Show that the set of all vectors X such that $AX = 5X$ is a subspace of R^n.

Let W be the set of all vectors X such that $AX = 5X$.

Let $X_1, X_2 \in W$. Then $AX_1 = 5X_1$ and $AX_2 = 5X_2$.

Consider $A(X_1 + X_2) = AX_1 + AX_2 = 5X_1 + 5X_2 = 5(X_1 + X_2)$

$\Rightarrow X_1 + X_2 \in W$.

Next, if c is a scalar and $X \in W$, then

$A(cX) = c(AX) = c(5X) = (c5)X = (5c)X = 5(cX)$

Hence $cX \in W$.

This shows that W is a subspace of R^n.

6. Find a basis and the dimension of the solution space of the system.

$$\begin{aligned} x_1 + 5x_2 + 13x_3 + 14x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \\ 2x_1 + 7x_2 + 17x_3 + 19x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 2 & 5 & 11 & 12 & 0 \\ 2 & 7 & 17 & 19 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc} 1 & 5 & 13 & 14 \\ 0 & -5 & -15 & -16 \\ 0 & -3 & -9 & -9 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[-5R_3+R_1]{5R_3+R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$\Rightarrow x_4 = 0, \quad x_3 = t, \quad x_2 = -3t, \quad x_1 = 2t$

$\therefore$ the solution to the system is $\begin{bmatrix} 2t \\ -3t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Hence $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the solution space of the system,

and clearly this solution space is one-dimensional.

7. Write True or False for each of the following:

(a) An $n \times n$ matrix B is invertible if and only if $det\, B \neq 0$. (✓)

(b) The system $AX = B$ has a unique solution if and only if A is non-singular.

(✓)

(c) If v_1, v_2, v_3, v_4, v_5 are linearly independent, then they are distinct. (✓)

(d) Two basis for a vector space must have the same number of vectors.(✓)

(e) If v_1, v_2, v_3, v_4, v_5 are vectors in R^4, then they are linearly independent.

(X)

(f) The matrix $\begin{bmatrix} 0 & -5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix. (✓)

(g) If $T \subset S$ and S is linearly dependent then T is linearly dependent.(X)

(h) Any set contains the zero vector must be linearly dependent. (✓)