KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

MATH 260-(032)

First Major Exam

March 20, 2004

Time: 90 Minutes

I.D.: Name: Solution Sec.# Serial #

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points	
1	/20	12+8
2	/18	6+6+6
3	/27	9+9+9
4	/15	10+5
Total	/80	

Instructor: M. Samman

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1.

12 (a) **Check** the following 1st order ODEs and write in the brackets *S* for *Separable*, *L* for *Linear*, *E* for *Exact*, *H* for *Homogeneous*, and *B* for *Bernoulli*:

i. $(e^{y/x}+e^{x^3/y^3}+1)dy=(1+\ln(y/x))dx$ (H)

ii. $(x^2y-y^3)dx=x^3dy$ (H) , B

iii. $x^y\ln x dy=-yx^{y-1}dx$ (S) , E

iv. $u^2+u=x^2u'$ (S) , B

v. $(y-xy^2)dy=ydx$ (L)

vi. $(x^2-4x-3xy+4)dx=(x^2-x-2)dy$ (L)

8 (b) A body whose temperature T is initially at $160^\circ c$ is immersed in a liquid whose temperature is kept constant at $100^\circ c$. If the temperature of the body is $130^\circ c$ at $t=1$ minute, what is its temperature at $t=2$ minutes? (<u>Your answer should not include logarithms</u>)

T = body temp.
T_m = liquid temp.
$\Rightarrow$ $T(0)=160$, $T(1)=130$
$T_m(0)=T_m=100$, $T(2)=?$

$$\frac{dT}{dt}=K(T-T_m)$$

2 $T=T_m+Ce^{kt}$

$T(0)=T_m(0)+C=160$

$\Rightarrow 100+C=160 \Rightarrow$ 2 $\boxed{C=60}$

$\therefore T=T_m+60e^{kt}$

NOW, $T(1)=130$, $T_m(1)=100$ const.

$T(1)=T_m(1)+60e^K=130$

$100+60e^K=130$

$\Rightarrow e^K=\frac{1}{2} \Rightarrow$ 2 $\boxed{K=\ln\frac{1}{2}}$

$\therefore T=T_m+60e^{(\ln\frac{1}{2})t}$

$T(2)=100+60e^{(\ln\frac{1}{2})2}=100+60e^{2(\ln\frac{1}{2})}=100+60e^{\ln\frac{1}{4}}=100+60(\frac{1}{4})$

2 $\therefore T(2)=115$

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6 **2.** (a) Convert (without solving) the **Bernoulli D.E.** $x^2dy = \left((\cos x)y - 2\sqrt{y}\ln x\right)dx$ into **linear D.E.**

$$\frac{dy}{dx} = \frac{(\cos x)y - 2\sqrt{y}\ln x}{x^2}$$

$$\frac{dy}{dx} - \left(\frac{\cos x}{x^2}\right)y = -\frac{2\ln x}{x^2}\, y^{\frac{1}{2}} \quad \text{------} \quad (*)$$

Let $w = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \Rightarrow y = w^2 \Rightarrow \frac{dy}{dx} = 2w\frac{dw}{dx}$

$(*)$ becomes

$$2w\frac{dw}{dx} - \left(\frac{\cos x}{x^2}\right)w^2 = -\frac{2\ln x}{x^2}\, w$$

$$\frac{dw}{dx} - \left(\frac{\cos x}{2x^2}\right)w = -\frac{\ln x}{x^2}$$

6 (b) **Only** find the "Integrating Factor" for the linear D.E. $(\csc x)y' - xy = \tan^2 x$.

$$y' - \frac{x}{\csc x}y = \frac{\tan^2 x}{\csc x}$$

$$\rho = e^{-\int \frac{x}{\csc x}dx} = e^{-\int x\sin x\, dx}$$. Using integration by parts :

Let $u = x$, $dv = \sin x\, dx$; $du = dx$, $v = -\cos x$ $\Big\}$ $\int u\,dv = uv - \int v\,du$

$$\therefore \int x\sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x$$

$$\therefore \rho = e^{x\cos x - \sin x}$$

6 (c) If $2y^2 - \sin x = c$ is a solution of an IVP $y' = f(x, y), y\left(\frac{\pi}{6}\right) = 2$ then find c.

$y(\pi/6) = 2$ i.e. $y = 2$ when $x = \pi/6$

$$\Rightarrow 2(4) - \sin\pi/6 = c$$

$$8 - \tfrac{1}{2} = c$$

$$c = \frac{15}{2}$$

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3. Solve each of the following DEs:

(a) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$

$$\underbrace{(2xy - \sec^2 x)}_{M}dx + \underbrace{(x^2 + 2y)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x. \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{the given DE is exact}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = M = 2xy - \sec^2 x \\ \dfrac{\partial f}{\partial y} = N = x^2 + 2y \end{cases}$$

$$f(x,y) = \int (2xy - \sec^2 x)\,dx$$

$$= x^2 y - \tan x + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = N = x^2 + 2y$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = y^2 + C_1$$

$$f(x,y) = x^2 y - \tan x + y^2 + C_1$$

The Solution is:

$$x^2 y - \tan x + y^2 = C$$

(b) $x^2 y' + y^2 = 0$. Is $y(x) = 0$ a singular solution ? Justify.

[You may solve also as a homogeneous DE

$$y' = -\frac{y^2}{x^2}$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$

$$\frac{dy}{y^2} = -\frac{dx}{x^2}$$

$$\frac{1}{y} = -\frac{1}{x} + C$$

or $\quad x + y = Cxy \qquad -- \qquad (*)$

$y(x) = 0$ is a Singular Solution because we cannot obtain it from the given one-parameter family of Solutions. i.e. there is no C in $(*)$ that will give this solution.

(c) $(x+2y-1)dx+3(x+2y)dy=0$

2 $$\left.\begin{aligned} \text{Let } u &= x+2y \\ du &= dx+2dy \\ dx &= du-2dy \end{aligned}\right\}$$

Substitute for x, dx in the given DE,

$$(u-1)(du-2dy)+3u\,dy=0$$

$$(u-1)du+(2-2u+3u)dy=0$$

$$(u-1)du+(u+2)dy=0$$

2 $$\frac{u-1}{u+2}du+dy=0$$

2 $$\left(1-\frac{3}{2+u}\right)du+dy=0 \quad , \text{ (using partial fractions)}$$

Integrating now, we get

1 $$u-3\ln|u+2|+y=C$$

$$x+2y-3\ln|x+2y+2|+y=C$$

1 $$x+3y-3\ln|x+2y+2|=C$$

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4. (a) Solve the following system:
$$\begin{aligned} 2x_1+3x_2+3x_3-x_4&=3\\ 5x_1+7x_2+4x_3+x_4&=5\\ x_1+x_2-2x_3+3x_4&=4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3\\ 5 & 7 & 4 & 1 & 5\\ 1 & 1 & -2 & 3 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4\\ 5 & 7 & 4 & 1 & 5\\ 2 & 3 & 3 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{-5R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4\\ 0 & 2 & 14 & -14 & -15\\ 0 & 1 & 7 & -7 & -5 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4\\ 0 & 1 & 7 & -7 & -15/2\\ 0 & 1 & 7 & -7 & -5 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4\\ 0 & 1 & 7 & -7 & -15/2\\ 0 & 0 & 0 & 0 & 5/2 \end{array}\right]$$

$\Rightarrow$ the system has no solution.

i.e. the system is inconsistent.

5 (b) A matrix A is called idempotent if $A^2 = A$. Determine all values of x, y such that the matrix $A = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$ is idempotent.

$$A^2 = A \implies \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$$

$$\begin{bmatrix} x^2-2 & x+y \\ -2x-2y & -2+y^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$$

$$\implies \begin{cases} x^2-2 = x \\ x+y = 1 \\ -2x-2y = -2 \\ -2+y^2 = y \end{cases}$$

Solving this system:

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$\Rightarrow \quad x = -1,\ 2$$

$$\Rightarrow \quad y = 2,\ -1$$

Hence the required values are: $x = -1$, $y = 2$

or $x = 2$, $y = -1$