KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

MATH 202-(031)

Second Major Exam

December 4, 2003

Time: 70 Minutes

I.D.: Name: Solution Sec.#

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points
1	/14
2	/16
3	/10
4	/10
5	/10
Total	/60

Instructor: M. Samman

1. (a) Show that e^x, e^{2x} and e^{3x} form a fundamental set of solutions for the DE $y''' - 6y'' + 11y' - 6y = 0$. Then write the general solution of this equation.

(5 points)

Consider first $y = e^x$. Then $y' = y'' = y''' = e^x$.

Substituting in the given DE, we get

$$y''' - 6y'' + 11y' - 6y = e^x - 6e^x + 11e^x - 6e^x = 0.$$

Hence $y = e^x$ is a solution for the given DE.

Similarly we can verify that both of e^{2x} and e^{3x} are solutions.

Now, to show that these solutions form a fundamental set, we have to show that they are linearly independent.

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x}(2)$$

$$= 2e^{6x}$$

$$\neq 0$$

Hence, e^x, e^{2x} and e^{3x} form a fundamental set of solutions.

(b) If $y_{p_1} = -\frac{1}{4}e^{-2x}$ is a particular solution of the DE $y'' + 2y' - 8y = 2e^{-2x}$, and $y_{p_2} = \frac{1}{9}e^{-x}$ is a particular solution of $y'' + 2y' - 8y = -e^{-x}$, obtain a solution for the DE $y'' + 2y' - 8y = -8e^{-2x} - 2e^{-x}$. (5 points)

Given;

$y_{p_1} = -\frac{1}{4}e^{-2x}$ is a particular solution of $y'' + 2y' - 8y = 2e^{-2x}$ —(1)

$y_{p_2} = \frac{1}{9}e^{-x}$ " " " " $y'' + 2y' - 8y = -e^{-x}$ —(2)

To obtain a solution for the DE $y'' + 2y' - 8y = -8e^{-2x} - 2e^{-x}$, —(*)

we observe that the L.H.S. of this equation is the same as the L.H.S. of the above two equations, and the R.H.S. of (*) is the sum of $-4(\text{R.H.S. of (1)}) + 2(\text{R.H.S. of (2)})$, which is

$$-4(2e^{-2x}) + 2(-e^{-x}) = -8e^{-2x} - 2e^{-x}$$

Hence, by the superposition principle, we deduce that

$$y = -4\left(-\tfrac{1}{4}\right)e^{-2x} + 2\left(\tfrac{1}{9}\right)e^{-x} \text{ is a solution of } (*)$$

i.e. $y = e^{-2x} + \frac{2}{9}e^{-x}$

(c) Find the most suitable annihilator for $x^2e^x \sin x - 5xe^x \cos x$ (4 points)

Note that an annihilator which annihilates $x^2e^x \sin x$ will also annihilate $-5xe^x \cos x$.

Identify $x^2e^x \sin x$ with $x^n e^{\alpha x} \sin \beta x$. Hence the annihilator is

$\alpha = 1, \beta = 1$

$(D^2 - 2D + 2)^3$.

2. Solve the following DEs

(a) $y'''' - 3y'' + 4y = 0$ **(8 points)**

The auxiliary equation is $\lambda^3 - 3\lambda^2 + 4 = 0$

$$\Rightarrow (\lambda+1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda+1)(\lambda-2)^2 = 0$$

$$\lambda = -1, 2, 2.$$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\therefore$ the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}$$

(b) $y'' + 4y' + 9y = 0$ **(8 points)**

The auxiliary equation is $\lambda^2 + 4\lambda + 9 = 0$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-36}}{2} = -2 \pm \sqrt{5}\, i$$

$$\downarrow$$

$$\alpha = -2, \beta = \sqrt{5}$$

Hence the solution is

$$y = e^{-2x}\left[c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x\right].$$

3. Use the method of undetermined coefficients to solve the DE $y''' + y'' = 8x^2$

(10 points)

First we solve the associated homogeneous equation; $y''' + y'' = 0$

$$\lambda^3 + \lambda^2 = 0 \Rightarrow \lambda^2(\lambda+1) = 0 \Rightarrow \lambda = 0, 0, -1$$

$$\therefore y_H = C_1 + C_2 x + C_3 e^{-x} \quad \text{------} \quad (1)$$

Now write the given non-homogeneous equation as:

$$(D^3 + D^2) y = 8x^2 . \quad \text{Note that } Ann(8x^2) = D^3.$$

$$D^3 (D^3 + D^2) y = D^3 (8x^2) = 0$$

$$D^5 (D+1) y = 0 \Rightarrow D^5 (D+1) = 0$$

$$\therefore \lambda^5 (\lambda+1) = 0 \Rightarrow \lambda = 0, 0, 0, 0, 0, -1$$

$$y = \cancel{c_1} + \cancel{c_2} x + \cancel{c_3} x^2 + \cancel{c_4} x^3 + \cancel{c_5} x^4 + \cancel{c_6} e^{-x}$$

or

$$y = \underbrace{c_1 + c_2 x + c_3 e^{-x}}_{y_H} + \underbrace{Ax^2 + Bx^3 + Cx^4}_{y_p} \quad \text{----} \quad (2)$$

Comparing (1) & (2), we get, $y_p = Ax^2 + Bx^3 + Cx^4$.

$$\Rightarrow y_p' = 2Ax + 3Bx^2 + 4Cx^3 , \quad y_p'' = 2A + 6Bx + 12Cx^2 , \quad y_p''' = 6B + 24Cx$$

Substituting in the given DE,

$$2A + 6Bx + 12Cx^2 + 6B + 24Cx = 8x^2$$

$$(2A + 6B) + (6B + 24C)x + 12Cx^2 = 8x^2$$

$$\Rightarrow \left.\begin{array}{l} 2A + 6B = 0 \\ 6B + 24C = 0 \\ 12C = 8 \end{array}\right\} \Rightarrow \text{Solving} \Rightarrow A = 8 , B = -\tfrac{8}{3} , C = \tfrac{2}{3}$$

$$\therefore y_p = 8x^2 - \tfrac{8}{3}x^3 + \tfrac{2}{3}x^4$$

So $y = y_H + y_p = c_1 + c_2 x + c_3 e^{-x} + 8x^2 - \frac{8}{3}x^3 + \frac{2}{3}x^4$

4. Use variation of parameters to solve $y'' + y = \sec x$ (10 points)

First we solve the associated hom. equation $y'' + y = 0$.

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \qquad [\alpha = 0, \beta = 1]$$

$$y_H = C_1 \cos x + C_2 \sin x \qquad \text{. so } \begin{array}{l} y_1 = \cos x \\ y_2 = \sin x \end{array}$$

We are seeking a particular solution for the given DE; $y_p = u_1 \cos x + u_2 \sin x$.

$$W = W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$u_1' = \frac{W_1}{W} = -\tan x$$

$$u_2' = \frac{W_2}{W} = 1$$

$$u_1 = \int -\tan x \, dx = \ln|\cos x|$$

$$u_2 = x$$

$$\therefore y_p = u_1 \cos x + u_2 \sin x$$

$$= \cos x \ln|\cos x| + x \sin x$$

$$y = y_H + y_p = C_1 \cos x + C_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

5. Solve the DE $x^2y'' + 3xy' + y = 0$ (10 points)

This is a Cauchy-Euler equation.

Let $y = x^m$. Then $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Substituting in the DE, we get

$$x^2y'' + 3xy' + y = x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + x^m = 0$$

$$\Rightarrow \quad m(m-1)x^m + 3mx^m + x^m = 0$$

$$[m(m-1) + 3m + 1]\, x^m = 0$$

$$(m^2 - m + 3m + 1)\, x^m = 0$$

$$(m^2 + 2m + 1)\, x^m = 0$$

$$(m+1)^2 x^m = 0$$

$$(m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$$

$\therefore$ the solution is $y = c_1 x^{-1} + c_2 x^{-1} \ln x$.