KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

MATH 202-(031)

First Major Exam

October 27, 2003

Time: 90 Minutes

I.D.: Name: Solution Sec.# Serial #

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points
1	/7
2	/15
3	/10
4	/12
5	/10
6	/16
Total	/70

Instructor: M. Samman

1. Consider the autonomous first-order differential equation $\frac{dy}{dx} = 3 - y$

(a) Find the critical point(s) of the given equation.

1

$$\frac{dy}{dx} = f(y) = 0$$

$\Rightarrow 3 - y = 0$

$\Rightarrow = 3$ is the critical point.

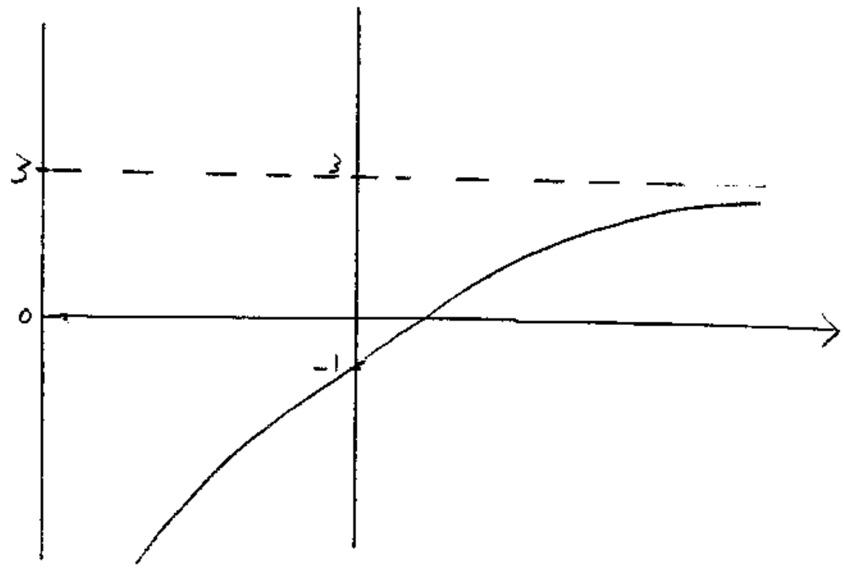
(b) Discuss the stability at the point(s) in (a).

3

Asymptotically stable at $x = 3$

(c) Given the initial condition $y(0) = -1$, sketch the graph of the solution.

3



7

2. (a) For each of the following, state whether the equation is linear or nonlinear, and give its order.

Equation	Linearity	Order
$y''' - 3y' + 2y = 0$	linear	3
$x(y'')^3 + (y')^4 - y = 0$	Non-linear	2
$y' = 1 - xy + y^2$	Non-linear	1

(b) Identify the following 1st Order ODE as **Separable, Linear in** y (or in x), **Homogeneous (with its degree), Bernoulli,** or **Exact**. Also write the ODE in the **standard form** of the identified category. (Write "**None**" if the ODE is not in any of the above-mentioned categories).

i. $(y + y^2)dx - (x + x^2)dy = 0$

$$\frac{dy}{dx} = \frac{y+y^2}{x+x^2} \leftarrow \text{Separable}$$

$$\frac{dy}{dx} = \frac{1}{x+x^2}(y+y^2) = f(x)\,g(y)$$

ii. $(y - xy^2)dy = ydx$

$$\frac{dx}{dy} - \frac{(y-xy^2)}{y} = 0$$

$$\frac{dx}{dy} - 1 + xy = 0$$

$$\frac{dx}{dy} + yx = 1 \leftarrow \text{linear in } x, \text{ of the form } \frac{dx}{dy} + P(y)x = f(y)$$

iii. $(e^{y/x} + e^{x^3/y^3} + 1)dy = (1 + \ln(y/x))dx$

$$\left(1 + \ln\left(\tfrac{y}{x}\right)\right)dx + \left(-e^{\frac{y}{x}} - e^{\frac{x^3}{y^3}} - 1\right)dy = 0$$

$$M\,dx + N\,dy = 0$$

$$M(tx,ty) = 1 + \ln\left(\tfrac{ty}{tx}\right) = 1 + \ln\left(\tfrac{y}{x}\right) = M(x,y) = t^0 M(x,y)$$

$$N(tx,ty) = \text{- - - - - - -} = t^0 N(x,y)$$

$\therefore$ Homogeneous of degree 0.

iv. $\frac{dy}{dx} = \sqrt{x^2 - y^2}$

None.

15

3. x^{-4} is the integrating factor of the linear ODE $xy' - 4y = x^6 e^x$

5 **(a)** Find the solution of this equation subject to the condition $y(1) = 5$.

$$xy' - 4y = x^6 e^x$$
$$y' - \frac{4}{x} y = x^5 e^x$$

Multiplying both sides by the integrating factor; x^{-4}, we get

$$x^{-4}\left[y' - \frac{4}{x} y\right] = x^{-4}\left[x^5 e^x\right]$$

$$\frac{d}{dx}\left[y x^{-4}\right] = x e^x$$

$$y x^{-4} = \int x e^x \, dx$$

$$y x^{-4} = x e^x - e^x + c$$

$$y = x^4\left[x e^x - e^x + c\right]$$

Since $y(1) = 5$, we have

$$5 = 1\left[e - e + c\right] \Rightarrow c = 5$$

$\therefore$ the solution is

$$y = x^4\left(x e^x - e^x + 5\right).$$

1+3 4 **(b)** Does the solution of the above DE subject to the condition $y(0) = 5$ exist ? Justify your answer.

No it does not. Note that at $x=0$, the coefficient of y' is zero and the condition of the existence theorem is not satisfied.

10

6

4. (a) Show that $\frac{1}{e^{x^2}}+\frac{1}{y^2}=2$ is a solution of the differential equation

$$e^{x^2}\frac{dy}{dx}+xy^3=0$$

$$e^{-x^2}+y^{-2}=2$$

using implicit differentiation, $-2xe^{-x^2}-2y^{-3}\frac{dy}{dx}=0$

$$\Rightarrow 2y^{-3}\frac{dy}{dx}=-2xe^{-x^2}$$

$$\frac{dy}{dx}=-xy^3e^{-x^2}$$

Substituting in the DE, we get $e^{x^2}\frac{dy}{dx}+xy^3=e^{x^2}(-xy^3e^{-x^2})+xy^3$

$$=-xy^3+xy^3$$

$$=0$$

∴ the DE is satisfied by the given solution.

(b) Solve the following differential equation: $\frac{dy}{dx}=\frac{y^2+xy^2}{x^2y-x^2}$

6

$$\frac{dy}{dx}=\frac{y^2(1+x)}{x^2(y-1)}$$

$$x^2(y-1)dy=y^2(1+x)dx$$

$$\frac{(y-1)}{y^2}dy=\frac{(1+x)}{x^2}dx$$

$$\int\quad=\int$$

$$\int\left(\frac{1}{y}-\frac{1}{y^2}\right)dy=\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx$$

$$\ln|y|+\frac{1}{y}=-\frac{1}{x}+\ln|x|+C$$

$$\ln\left(\frac{y}{x}\right)+\frac{1}{y}+\frac{1}{x}=C$$

12

5. Show that the differential equation $6y^2dx - x^3(2x + \frac{y}{x^2})dy = 0$ is Bernoulli equation, and hence find its particular solution subject to condition $y(-1) = 1$.

$$6y^2 dx - x^3\left(2x + \frac{y}{x^2}\right)dy = 0$$

$$\frac{dx}{dy} + \frac{(-2x^4 - xy)}{6y^2} = 0$$

$$\frac{dx}{dy} - \frac{x^4}{3y^2} - \frac{x}{6y} = 0$$

$$(*) \text{ --- } \frac{dx}{dy} - \frac{1}{6y}x = \frac{1}{3y^2}x^4$$ which is Bernoulli eqⁿ of the form

$$\frac{dx}{dy} + P(y)x = f(y)x^n$$

Put $W = x^{1-n} = x^{1-4} = x^{-3} \Rightarrow x = w^{-\frac{1}{3}},\ x^4 = w^{-\frac{4}{3}}$

$$\frac{dw}{dy} = -3x^{-4}\frac{dx}{dy} = -3\left(w^{-\frac{1}{3}}\right)^{-4}\frac{dx}{dy} = -3w^{\frac{4}{3}}\frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{1}{3}w^{-\frac{4}{3}}\frac{dw}{dy}$$

Substitute in $(*)$, we get

$$-\frac{1}{3}w^{-\frac{4}{3}}\frac{dw}{dy} + \left(-\frac{1}{6y}\right)w^{-\frac{1}{3}} = \frac{1}{3y^2}w^{-\frac{4}{3}}$$

$$\frac{dw}{dy} + \frac{1}{2y}w = -\frac{1}{y^2}$$ which is linear in W.

$$IF = e^{\int \frac{1}{2y}dy} = e^{\frac{1}{2}\ln|y|} = \sqrt{y} \quad , y > 0$$

$$\frac{d}{dy}\left[w\sqrt{y}\right] = \frac{-\sqrt{y}}{y^2} = -y^{-\frac{3}{2}}$$

$$w\sqrt{y} = 2y^{-\frac{1}{2}} + C$$

$$w = 2y^{-1} + Cy^{-\frac{1}{2}}$$

$$\therefore x^{-3} = 2y^{-1} + Cy^{-\frac{1}{2}}$$

$$\frac{1}{x^3} = \frac{2}{y} + \frac{C}{\sqrt{y}}$$

Now using the condition $y(-1) = 1$,

$$-1 = 2 + C \Rightarrow C = -3$$

$\therefore$ the solution is

$$\frac{1}{x^3} = \frac{2}{y} - \frac{3}{\sqrt{y}}$$

5

10

6

6. (a) Find the integrating factor of the linear differential equation

$$(1-x^2)dy = [xy + (1+x^2)x\ln x]dx$$

$$[xy + (1+x^2)x\ln x]dx + (x^2-1)dy = 0$$

$$M\,dx + N\,dy = 0$$

$\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2x$. It is not exact, so we find an integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x - 2x = -x. \Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x^2-1}(-x) = \frac{-x}{x^2-1} = f(x)$$

$$\therefore IF = e^{\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]dx} = e^{-\int \frac{x}{x^2-1}dx} = e^{-\frac{1}{2}\ln|x^2-1|} = (x^2-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2-1}}$$

10

(b) When a Cake is removed from an oven, its temperature is measured 350^0 F. After 3 minutes its temperature is 210^0 F. If the room temperature is 70^0 F, find Cake temperature after 6 minutes from the time it was taken from the oven.

$T :=$ Cake temp.
$T_m :=$ Room temp.
$\Rightarrow T(0) = 350$, $T(3) = 210$
$T_m(t) = 70$, $T(6) = ?$

$$\frac{dT}{dt} = K(T - T_m)$$

$$T = T_m + Ce^{Kt}$$

$$T(0) = T_m(0) + Ce^{K(0)}$$

$$350 = 70 + C \Rightarrow C = 280$$

$$\therefore T = T_m + 280e^{Kt}$$

$$T(3) = 70 + 280e^{3K} = 210 \Rightarrow 280e^{3K} = 140 \Rightarrow e^{3K} = \frac{1}{2}$$

$$\therefore 3K = \ln\frac{1}{2} \Rightarrow K = \frac{1}{3}\ln\frac{1}{2}$$

Now $T(t) = 70 + 280e^{(\frac{1}{3}\ln\frac{1}{2})t}$

$$T(6) = 70 + 280e^{(\frac{1}{3}\ln\frac{1}{2})6} = 70 + 280e^{2\ln\frac{1}{2}} = 70 + 280\left(\frac{1}{4}\right)$$

$$\therefore T(6) = 70 + 70 = 140^\circ F$$

16