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KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

MATH 202-(031)

First Major Exam

October 9, 2003

Time: 90 Minutes

I.D.: Name: Solution Serial #

No Calculator is Allowed in the Exam

Show All Necessary Work

Question	Points
1	/8
2	/14
3	/10
4	/10
5	/10
6	/10
7	/10
8	/8
Total	/80

Instructor: M. Samman

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1. Consider the autonomous first-order differential equation $\frac{dy}{dx} = 10 + 3y - y^2$

2 **(a)** Find the critical points of the given equation.

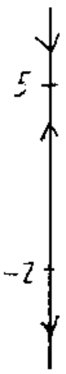
$$\frac{dy}{dx} = f(y) = 0$$

$$10 + 3y - y^2 = 0$$

$$(5 - y)(2 + y) = 0$$

$y = -2, 5$ the critical points.

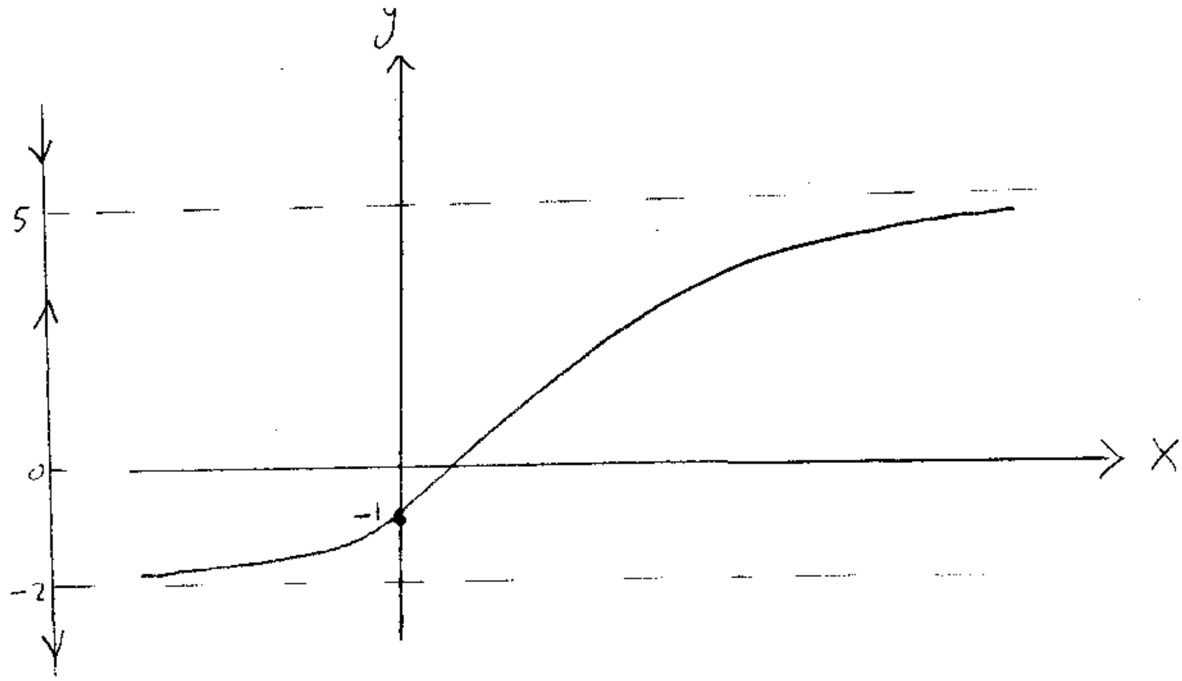
3 **(b)** Discuss the stability at each point in (a).



stable at $y = 5$

unstable at $y = -2$

3 **(c)** Given the initial condition $y(0) = -1$, sketch the graph of the solution.



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(3) **2. (a)** For each of the following, state whether the equation is linear or nonlinear, and give its order.

	Equation	Linearity	Order
1	$y''' - 3y' + 2y = 0$	linear	3
1	$x(y'')^3 + (y')^4 - y = 0$	Non-linear	2
1	$y' = 1 - xy + y^2$	Non-linear	1

(b) Identify the following 1st Order ODE as **Separable, Linear in** y (or **in** x), **Homogeneous (with its degree), Bernoulli,** or **Exact**. Also write the ODE in the **standard form** of the identified category. (Write "**None**" if the ODE is not in any of the above-mentioned categories).

(3) **i.** $(3y^2 + y + 3x)dx = (4 - 6xy - x)dy$

$(3y^2+y+3x) + (-4+6xy+x)dy = 0$

$\frac{\partial M}{\partial y} = 6y+1$, $\frac{\partial N}{\partial x} = 6y+1 \Rightarrow$ Exact

$Mdx + Ndy = 0$

(3) **ii.** $3\frac{dy}{dx} = 4x - y$

$\frac{dy}{dx} + \frac{1}{3}y = \frac{4}{3}x \Rightarrow$ Linear

$\frac{dy}{dx} + P(x)y = g(x)$

(3) **iii.** $(e^{y/x} + e^{x^3/y^3} + 1)dy = (1 + \ln(y/x))dx$

$1 + \ln(y/x))dx + (-e^{y/x} - e^{x^3/y^3} - 1)dy = 0$

$M(tx,ty) = 1 + \ln\left(\frac{ty}{tx}\right) = 1 + \ln\left(\frac{y}{x}\right) = M(x,y) = t^0 M(x,y)$

$N(tx,ty) =$ — — — — $= t^0 M(x,y)$

∴ Homogeneous of degree 0.

(2) **iv.** $(y + y^2)dx - (x + x^2)dy = 0$

$\frac{dy}{y+y^2} = \frac{dx}{x+x^2} \Longrightarrow$ separable

$\frac{dy}{H(y)} = \frac{dx}{M(x)}$ or $\frac{dy}{dx} = f(x)g(y)$

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3. (a) It is known that $y = \dfrac{1+ce^{2x}}{1+ce^{-2x}}$ is a one parameter family of Solutions of the ODE

4 $y' = y^2 - 1$. Find a Singular Solution of this ODE.

$y' = y^2 - 1$

$= (y-1)(y+1) \Rightarrow y = 1, -1$ are solutions

Note that $y = -1$ is a solution that can not be obtained from the above family of solutions. So $y = -1$ is a Singular solution.

(b) Show that $\dfrac{1}{e^{x^2}} + \dfrac{1}{y^2} = 2$ is a solution of the differential equation $e^{x^2}\dfrac{dy}{dx} + xy^3 = 0$

6 $e^{-x^2} + y^{-2} = 2$

using implicit differentiation

$-2xe^{-x^2} - 2y^{-3}\dfrac{dy}{dx} = 0$

$2y^{-3}\dfrac{dy}{dx} = -2xe^{-x^2}$

$\dfrac{dy}{dx} = -xy^3e^{-x^2}$

Substituting in the DE,

$$e^{x^2}\frac{dy}{dx} + xy^3 = e^{x^2}\left(-xy^3e^{-x^2}\right) + xy^3$$

$$= -xy^3 + xy^3$$

$$= 0$$

$\longrightarrow$

$\therefore$ the DE is satisfied by the given solution.

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4. Show that the differential equation $6x^2dy - y^3(2y + \frac{x}{y^2})dx = 0$ is Bernoulli equation, and hence find its particular solution subject to condition $y(1) = -1$.

$$6x^2 dy - y^3\left(2y + \frac{x}{y^2}\right)dx = 0$$

$$6x^2 dy + (-2y^4 - xy)dx = 0$$

$$\frac{dy}{dx} + \frac{(-2y^4 - xy)}{6x^2} = 0$$

$$\frac{dy}{dx} - \frac{y^4}{3x^2} - \frac{y}{6x} = 0$$

$(*)$ --- $\frac{dy}{dx} + \left(\frac{-1}{6x}\right)y = \frac{1}{3x^2}y^4$ Bernolli, of the form $\frac{dy}{dx} + P(x)y = f(x)y^n$

Put $w = y^{1-n} = y^{1-4} = y^{-3} \Rightarrow y = w^{\frac{-1}{3}}$, $y^4 = w^{\frac{-4}{3}}$

$$\frac{dw}{dx} = -3y^{-4}\frac{dy}{dx} = -3w^{\frac{4}{3}}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{3}w^{\frac{-4}{3}}\frac{dw}{dx}$$

Substituting in $(*)$,

$$-\frac{1}{3}w^{\frac{-4}{3}}\frac{dw}{dx} + \left(\frac{-1}{6x}\right)w^{\frac{-1}{3}} = \frac{1}{3x^2}w^{\frac{-4}{3}}$$

$$\frac{dw}{dx} + \frac{1}{2x}w = -\frac{1}{x^2}$$

$$IF = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln|x|} = \sqrt{x}, \quad x > 0$$

$$\frac{d}{dx}\left[w\sqrt{x}\right] = -\frac{\sqrt{x}}{x^2} = -x^{-\frac{3}{2}}$$

$$w\sqrt{x} = 2x^{-\frac{1}{2}} + C$$

$$w = 2x^{-1} + Cx^{-\frac{1}{2}}$$

$$y^{-3} = 2x^{-1} + Cx^{-\frac{1}{2}}$$

$$\frac{1}{y^3} = \frac{2}{x} + \frac{c}{\sqrt{x}}$$

$\therefore$ the solution is

$$\frac{1}{y^3} = \frac{2}{x} + \frac{c}{\sqrt{x}}$$

$y(1) = -1 \Rightarrow -1 = 2 + C$

$\Rightarrow C = -3$

$\therefore$ the solution is

$$\frac{1}{y^3} = \frac{2}{x} - \frac{3}{\sqrt{x}}$$

5. Use appropriate substitution in order to convert the following ODE to Separable, and then find its solution: $(x^2 + y^2)dx - xydy = 0$

$$(x^2+y^2)dx - xy\,dy = 0$$

Let $y = ux$

$dy = u\,dx + x\,du$

substituting,

$$(x^2+u^2x^2)dx - ux^2(u\,dx + x\,du) = 0$$

$$(x^2+\cancel{u^2x^2}-\cancel{u^2x^2})dx - ux^3du = 0$$

$$\frac{dx}{x} = u\,du$$

$$\ln|x| = \frac{1}{2}u^2 + C$$

$$\ln|x| = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C$$

6. Solve the following differential equation: $\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2 y - x^2}$

$$\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2 y - x^2}$$

$$\frac{dy}{dx} = \frac{y^2(1+x)}{x^2(y-1)}$$

$$x^2(y-1)\,dy = y^2(1+x)\,dx$$

$$\frac{(y-1)}{y^2}\,dy = \frac{(1+x)}{x^2}\,dx$$

$$\int \frac{(y-1)}{y^2}\,dy = \int \frac{(1+x)}{x^2}\,dx$$

$$\int\left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$$

$$\ln|y| + \frac{1}{y} = -\frac{1}{x} + \ln|x| + C$$

7. **Solve the following differential equation:** $(3x^2y+y^2)dx+(x^3+2xy)dy=0$

$$\frac{\partial M}{\partial y} = 3x^2+2y \,, \qquad \frac{\partial N}{\partial x} = 3x^2+2y \Rightarrow \text{exact}$$

$$f(x,y) = \int (3x^2y+y^2)\,dx = x^3y+xy^2+g(y)$$

$$\frac{\partial f}{\partial y} = x^3+2xy+g'(y) \qquad = N = x^3+2xy$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = C_1$$

$$f(x,y) = x^3y+xy^2+C_1$$

sol. $\quad x^3y+xy^2 = C.$

8. A body whose temperature T is initially at $160^{\circ}c$ is immersed in a liquid whose temperature is kept constant at $100^{\circ}c$. If the temperature of the body is $130^{\circ}c$ at $t=1$ minute, what is its temperature at $t=2$ minutes? (Your answer should not include logarithms)

$$\left.\begin{array}{l} T := \text{body temp} \\ T_m := \text{liquid temp} \end{array}\right\} \Rightarrow \begin{array}{l} T(0) = 160 \quad , \quad T(1) = 130 \\ T_m(0) = T_m = 100 \quad , \quad T(2) = ? \end{array}$$

$$\frac{dT}{dt} = K(T - T_m)$$

$$T = T_m + Ce^{Kt} \quad --- \quad (*)$$

$$T(0) = T_m(0) + Ce^{K(0)} = 160$$

$$\Rightarrow \quad 100 + C = 160$$

$$\Rightarrow \quad C = 60$$

$$\therefore \quad T = T_m + 60e^{Kt}$$

Now, $T(1) = 130$, $T_m(1) = 100$ Const.

$$T(1) = T_m(1) + 60e^{K} = 130$$

$$100 + 60e^{K} = 130$$

$$e^{K} = \tfrac{1}{2}$$

$$K = \ln\tfrac{1}{2}$$

$$\therefore \quad T = T_m + 60e^{(\ln\frac{1}{2})t}$$

$$T(2) = 100 + 60e^{(\ln\frac{1}{2})2} = 100 + 60e^{2(\ln\frac{1}{2})} = 100 + 60e^{\ln\frac{1}{4}}$$

$$T(2) = 100 + 60\left(\tfrac{1}{4}\right) = 100 + 15$$

$$\therefore T(2) = 115$$