Notes #1

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1 Bases in Banach Spaces

In this section X will denote a Banach space. The norm in X will be denoted by $\|\cdot\|$ and the dual of X will be denoted by X^*. A sequence $\{x_k\}_{k=1}^{\infty}$ in X is an ordered set. i.e.,

$$\{x_k\}_{k=1}^{\infty} = \{x_1, x_2, \cdots\}.$$

Definition 1 *The span of a sequence $\{e_k\}_{k=1}^{\infty}$ in X is the set of finite linear combinations of the elements of $\{e_k\}_{k=1}^{\infty}$.*

Definition 2 *A sequence $\{e_k\}_{k=1}^{\infty}$ in X is complete if $\overline{\operatorname{span}}\{e_k\}_{k=1}^{\infty} = X$.*

Definition 3 *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X.*

1. *The series $\sum_{k=1}^{\infty} x_k$ converges in X if the sequence of partial sums $s_n = \sum_{k=1}^{n} x_k$ converges in X. That is, if there is an $x \in X$ such that $\|s_n - s\| \to 0$ as $n \to \infty$.*

2. *The series $\sum_{k=1}^{\infty} x_k$ converges unconditionally in X if $\sum_{k=1}^{\infty} x_{\sigma(k)}$ converges for all permutations (1:1 and onto maps) $\sigma : \mathbb{N} \to \mathbb{N}$.*

Next we define the Schauder Bases in a Banach space.

Definition 4 *A sequence $\{e_k\}_{k=1}^{\infty}$ in X is a Schauder Basis for X if, for each $f \in X$, there exists a unique sequence of complex numbers $\{c_k\}_{k=1}^{\infty}$ (dependent on f) such that*

$$f = \sum_{k=1}^{\infty} c_k e_k. \tag{1}$$

In general, the convergence in the above definition is conditional, i.e., dependent on the choice of the order of the sequence $\{e_k\}_{k=1}^{\infty}$. If, however, the series (1) converges unconditionally to f for every $f \in X$, the basis $\{e_k\}_{k=1}^{\infty}$ is called an *unconditional basis.*

1.1 Uniqueness and Independence

The requirement that the series representation (1) be unique implies that the basis $\{e_k\}_{k=1}^{\infty}$ should be independent in a proper sense.

Definition 5 *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X.*

1. *$\{x_k\}_{k=1}^{\infty}$ is linearly independent if every finite subset of it is linearly independent.*
2. *$\{x_k\}_{k=1}^{\infty}$ is ω-independent if $\sum_{k=1}^{\infty} c_k x_k = 0 \Longrightarrow c_k = 0 \ \forall k \in \mathbb{N}$.*
3. *$\{x_k\}_{k=1}^{\infty}$ is minimal if $x_j \notin \overline{\text{span}}\,\{e_k\}_{k \neq j} \ \ \forall j \in \mathbb{N}$.*

We have the following relations between the above notions of independence.

Lemma 6

1. *$\{x_k\}_{k=1}^{\infty}$ minimal $\Longrightarrow$ $\{x_k\}_{k=1}^{\infty}$ ω-independent.*
2. *$\{x_k\}_{k=1}^{\infty}$ ω-independent $\Longrightarrow$ $\{x_k\}_{k=1}^{\infty}$ linearly independent.*

Proof. (Exercise). ■

Note that ω-independence $\Longrightarrow$ uniqueness of the representation.

1.2 A Characterization of Schauder Bases

The following characterization of a Schauder basis holds.

Theorem 7 *Suppose $\{e_k\}_{k=1}^{\infty}$ is complete in X. $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis for X if and only if there exists a constant $K > 0$ such that for all $m, n \in \mathbb{N}$, with $m \leq n$, and all sequences of complex numbers $\{c_k\}_{k=1}^{\infty}$*

$$\left\| \sum_{k=1}^{m} c_k e_k \right\| \leq K \left\| \sum_{k=1}^{n} c_k e_k \right\|. \tag{2}$$

Proof. (Sketch.) $\Longrightarrow$. Suppose $\{e_k\}_{k=1}^{\infty}$ is a basis. For $f \in X$ define

$$\|f\|_s = \sup_n \left\| \sum_{k=1}^{n} c_k e_k \right\|,$$

where $f = \sum_{k=1}^{\infty} c_k e_k$.

- $\|\cdot\|_s$ is a norm on X equivalent to the norm $\|\cdot\|$.(This follows from the Open Mapping Theorem)
- (2) follows by taking $f = \sum_{k=1}^{n} c_k e_k$.

$\Leftarrow$. Suppose $\{e_k\}_{k=1}^{\infty}$ satisfies (2). Define

$$\mathcal{A} = \left\{ f \in X : \; f = \sum_{k=1}^{\infty} c_k e_k \right\}.$$

- $\mathcal{A}$ is dense in X.
- $\mathcal{A}$ is closed in X :
  - Let $\{f_j\}$ be a sequence in $\mathcal{A}$ such that $f_j \to f$ in X. Write $f_j = \sum_{k=1}^{\infty} c_k^{(j)} e_k$. By (2) we can show that

    $$\begin{aligned} \left| c_i^{(j)} - c_i^{(l)} \right| \|e_i\| &\leq 2K \left\| \sum_{k=1}^{m} \left(c_k^{(j)} - c_k^{(l)} \right) e_k \right\|, \; m \geq i. \\ &= 2K \left\| \sum_{k=1}^{m} c_k^{(j)} e_k - \sum_{k=1}^{m} c_k^{(l)} e_k \right\|. \end{aligned}$$

    As $m \to \infty$,

    $$\left| c_i^{(j)} - c_i^{(l)} \right| \|e_i\| \leq 2K \|f_j - f_l\| .$$

    Therefore, $c_i^{(j)} \to c_i$ for each $i \in \mathbb{N}$.
  - Also, by (2),

    $$\left\| \sum_{k=1}^{m} \left(c_k^{(j)} - c_k^{(l)} \right) e_k \right\| \leq K \left\| \sum_{k=1}^{n} \left(c_k^{(j)} - c_k^{(l)} \right) e_k \right\|, \; n \geq m.$$

    As $n \to \infty$,

    $$\left\| \sum_{k=1}^{m} \left(c_k^{(j)} - c_k^{(l)} \right) e_k \right\| \leq K \|f_j - f_l\| .$$

    As $l \to \infty$,

    $$\left\| \sum_{k=1}^{m} \left(c_k^{(j)} - c_k \right) e_k \right\| \leq K \|f_j - f\| .$$

    As $m \to \infty$,

    $$\overline{\lim}_m \left\| f_j - \sum_{k=1}^{m} c_k e_k \right\| \leq K \|f_j - f\| .$$

    As $j \to \infty$,

    $$\overline{\lim}_m \left\| f - \sum_{k=1}^{m} c_k e_k \right\| = 0.$$

    Hence, $\sum_{k=1}^{\infty} c_k e_k$ exists and

    $$f = \sum_{k=1}^{\infty} c_k e_k.$$

– To show uniqueness of the presentation we show that $\{e_k\}_{k=1}^{\infty}$ is ω-independent. Assume

$$\sum_{k=1}^{\infty} c_k e_k = 0.$$

By (2),

$$|c_i| \|e_i\| \leq K \left\| \sum_{k=1}^{m} c_k e_k \right\| \to 0 \text{ as } m \to \infty.$$

■

Corollary 8 *Assume $\{e_k\}_{k=1}^{\infty}$ is a basis in X. The associated coefficient functionals: $l_k(f) = c_k$ are continuous linear functionals. Moreover, if $\|e_k\| \geq C$ for all $k \in \mathbb{N}$ then $\{l_k\}_{k=1}^{\infty}$ is uniformly bounded in X^*.*

Proof. (Exercise). ■

The smallest constant for which (2) holds is called the *basis constant* and is given by

$$K = \sup \left\{ \sum_{k=1}^{n} c_k e_k : m \leq n, \sum_{k=1}^{m} c_k e_k = 1 \right\}.$$

If $K = \infty$, then $\{e_k\}_{k=1}^{\infty}$ is not a basis.

Theorem 9 *Suppose $\{e_k\}_{k=1}^{\infty}$ is complete in X. $\{e_k\}_{k=1}^{\infty}$ is an unconditional basis if and only if*

$$\sup \left\{ \sum \sigma_k c_k e_k : \sum c_k e_k = 1, \sigma_k = \pm 1 \ \forall k \in \mathbb{N} \right\} < \infty.$$

Proof. See Literature. ■

Definition 10 *Two sequences $\{f_k\}_{k=1}^{\infty} \subset X$ and $\{g_k\}_{k=1}^{\infty} \subset X^*$ are called biorthogonal if $g_k(f_j) = \delta_{kj}$.*

It follows that a basis $\{e_k\}_{k=1}^{\infty}$ for X and the associated coefficient functionals $\{l_k\}_{k=1}^{\infty}$ are biorthogonal. We also have the following theorem.

Theorem 11 *Let $\{e_k\}_{k=1}^{\infty}$ be a basis for X and $\{l_k\}_{k=1}^{\infty}$ be the associated coefficient functionals. Then:*

1. *$\{l_k\}_{k=1}^{\infty}$ is a basis for its closed span in X^*. Moreover, $\{e_k\}_{k=1}^{\infty}$ considered as a sequence in X^{**} is its biorthogonal system.*

2. *If X is reflexive, then $\{l_k\}_{k=1}^{\infty}$ is a basis for X^*.*

Proof. See literature. ■

1.2.1 Bessel Sequences in Hilbert Spaces

From now on we deal with Hilbert spaces. H will denote a general Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Also, $\ell^2(\mathbb{N})$ will denote the Hilbert space of square summable sequences. Elements in $\ell^2(\mathbb{N})$ will be denoted by $c = (c_k), d = (d_k), \cdots$ etc.

Lemma 12 *Let $\{f_k\}_{k=1}^{\infty} \subset H$ be a sequence such that $\sum c_k f_k \in H$ for each $c = (c_k) \in \ell^2(\mathbb{N})$. The operator $T : \ell^2(\mathbb{N}) \to H$ defined by*

$$Tc = \sum_{k=1}^{\infty} c_k f_k$$

is linear and bounded. Its adjoint $T^* : H \to \ell^2(\mathbb{N})$ is defined by

$$T^* f = (\langle f, f_k \rangle).$$

Moreover,

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \le \|T\|^2 \|f\|^2. \tag{3}$$

Proof. Define $T_n : \ell^2(\mathbb{N}) \to H$ by

$$T_n c = \sum_{k=1}^{n} c_k f_k.$$

T_n is bounded and $T_n c \to Tc$ for each $c \in \ell^2(\mathbb{N})$. By the uniform boundedness principle, T is linear and bounded. To compute T^* observe first that

$$\langle Tc, f \rangle = \left\langle \sum_{k=1}^{\infty} c_k f_k, f \right\rangle = \sum_{k=1}^{\infty} c_k \langle f_k, f \rangle.$$

Hence, $\sum_{k=1}^{\infty} c_k \langle f_k, f \rangle$ converges for each $c \in \ell^2(\mathbb{N})$. Define the linear functionals $l_n : \ell^2(\mathbb{N}) \to \mathbb{C}$ by

$$l_n c = \sum_{k=1}^{n} c_k \langle f_k, f \rangle$$

and the functional $l : \ell^2(\mathbb{N}) \to \mathbb{C}$ by

$$lc = \sum_{k=1}^{\infty} c_k \langle f_k, f \rangle.$$

Then $l_n c \to lc$. Therefore, l is a bounded linear functional. By the Riesz Representation Theorem,

$$lc = \sum_{k=1}^{\infty} c_k \overline{a}_k$$

for some $a \in \ell^2(\mathbb{N})$. Thus

$$\sum_{k=1}^{\infty} c_k \langle f_k, f \rangle = \sum_{k=1}^{\infty} c_k \overline{a}_k.$$

Taking c to be the i^{th} standard basis element $\epsilon_i \in \ell^2(\mathbb{N})$, we get

$$\langle f_i, f \rangle = \overline{a}_i \ \forall i \in \mathbb{N}.$$

Therefore,

$$a = (\langle f, f_i \rangle) \in \ell^2(\mathbb{N}).$$

Now

$$\begin{aligned} \langle c, T^* f \rangle &= \langle Tc, f \rangle = \sum_{k=1}^{\infty} c_k \langle f_k, f \rangle \\ &= \langle c, (\langle f, f_i \rangle) \rangle_{\ell^2(\mathbb{N})}. \end{aligned}$$

Therefore,

$$T^* f = (\langle f, f_i \rangle).$$

Also,

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 &= \|T^* f\|^2 \leq \|T^*\|^2 \|f\|^2 \\ &= \|T\|^2 \|f\|^2. \end{aligned}$$

■

Inequality (3) plays a crucial role in the theory of frames. This is why we are going to pay closer attention to systems satisfying (3).

Definition 13 *A sequence $\{f_k\}_{k=1}^{\infty} \subset H$ is called a Bessel sequence if there exists a $B > 0$ such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2. \tag{4}$$

B is called a Bessel bound for $\{f_k\}_{k=1}^{\infty}$.

Theorem 14 *$\{f_k\}_{k=1}^{\infty} \subset H$ is a Bessel sequence with Bessel bound B if and only if the mapping*

$$c \mapsto \sum_{k=1}^{\infty} c_k f_k$$

defines a linear bounded operator T on $\ell^2(\mathbb{N})$ into H with $\|T\| \leq \sqrt{B}$.

Proof. $\Longrightarrow$. Suppose $\{f_k\}_{k=1}^{\infty}$is a Bessel sequence with Bessel bound B. We want to show that the mapping $c \mapsto \sum_{k=1}^{\infty} c_k f_k$ defines a bounded linear operator T on $\ell^2(\mathbb{N})$ into H. This will follow from Lemma 12 if we can show that $\sum_{k=1}^{\infty} c_k f_k \in H$ for every $c \in \ell^2(\mathbb{N})$. This can be done by showing that $\sum_{k=1}^{n} c_k f_k$ is a Cauchy sequence. Let $m > n$.

$$\begin{aligned}
\left\| \sum_{k=1}^{n} c_k f_k - \sum_{k=1}^{m} c_k f_k \right\| &= \left\| \sum_{k=m+1}^{n} c_k f_k \right\| \\
&= \sup_{\|g\|=1} \left| \left\langle \sum_{k=m+1}^{n} c_k f_k, g \right\rangle \right| \\
&= \sup_{\|g\|=1} \left| \sum_{k=m+1}^{n} c_k \langle f_k, g \rangle \right| \\
&\leq \sup_{\|g\|=1} \left(\sum_{k=m+1}^{n} |c_k|^2 \right)^{1/2} \left(\sum_{k=m+1}^{n} |\langle f_k, g \rangle|^2 \right)^{1/2} \\
&\leq \sqrt{B} \left(\sum_{k=m+1}^{n} |c_k|^2 \right)^{1/2}.
\end{aligned}$$

This establishes what we want since $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. To check that $\|T\| \leq \sqrt{B}$, we use Lemma 12 again to get

$$\|T^* f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 .$$

Hence, $\|T\| = \|T^*\| \leq \sqrt{B}$.

$\Longleftarrow$. This is the content of Lemma 12. ∎

It thus follows that to check that a sequence $\{f_k\}_{k=1}^{\infty} \subset H$ is a Bessel sequence, we only need to check that the operator T is well defined.

Since the Bessel condition (4) remains the same under permutations of the elements of $\{f_k\}_{k=1}^{\infty}$, we have the following corollary.

Corollary 15 *If $\{f_k\}_{k=1}^{\infty} \subset H$ is a Bessel sequence then $\sum_{k=1}^{\infty} c_k f_k$ converges unconditionally for all $c \in \ell^2(\mathbb{N})$.*