# 1 Intordution to Polynomial Interpolation

Although Taylor polynomials provide "best approximations" of functions $f$ defined on an interval $[a, b]$, they have several shortcomings:

1. To use a polynomial $P_n$ we have to know $n+1$ derivatives of $f$. These can either be unavailable or hard to evaluate.
2. They require the function $f$ to have $n+1$ continuous derivatives.
3. The approximation is local in nature in the sense that the approximation is good only near the point of evaluarion of the derivatives $x_0$.

An alternative is to construct a polynomial $P_n(x)$ of degree $n$ which shares $(n+1)$ points with the function $f$ :

$$P_n(x_k) = f(x_k), \ k = 0, 1, \ldots, n,$$

where $a \leq x_0 \leq x_1 \leq \ldots \leq x_n \leq b$. This way we do not need to evaluate derivatives of $f$ and we have the approximation over the whole interval $[a, b]$ rather than a single point $x_0$. The polynoial constructed this way is called the *interpolating polynomial* for the function $f$ at the points $x_0, x_1, \ldots, x_n$.

**Example** Suppose we want to construct the polynomial $P_5(x)$ of degree 5 which approximates the function $f(x) = \ln(1+x)$ over the intervale $[0, 1]$. We can make

$$P_5(x_k) = \ln(1 + x_k), \tag{1}$$

where

$$x_k \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}.$$

To construct this polynomial we write

$$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

and substitute in the equation (1) for $x = 0, 0.2, 0.4, 0.6, 0.8, 1$. This gives the linear system of equatoins

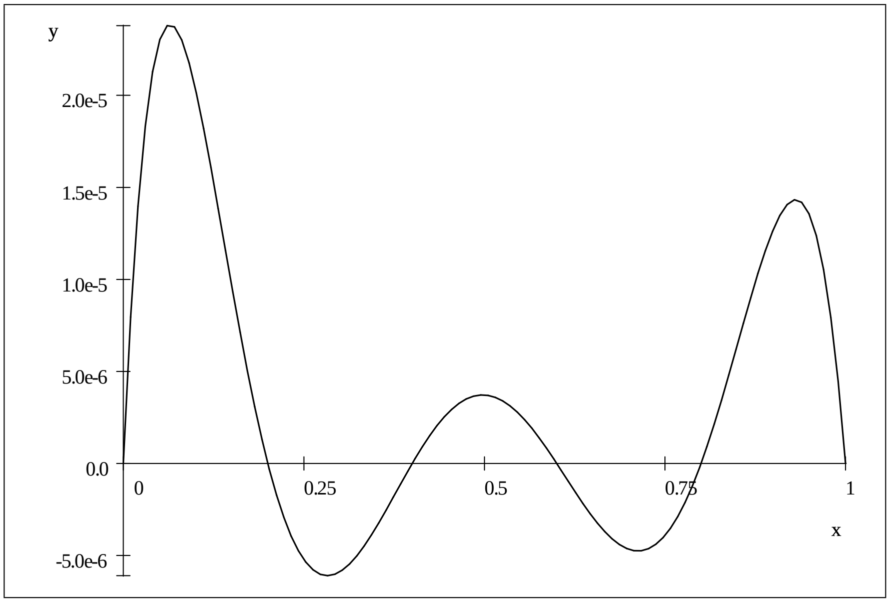
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & .2 & .2^2 & .2^3 & .2^4 & .2^5 \\ 1 & .4 & .4^2 & .4^3 & .4^4 & .4^5 \\ 1 & .6 & .6^2 & .6^3 & .6^4 & .6^5 \\ 1 & .8 & .8^2 & .8^3 & .8^4 & .8^5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} \ln 1 \\ \ln(1.2) \\ \ln(1.4) \\ \ln(1.6) \\ \ln(1.8) \\ \ln(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.182\,32 \\ 0.336\,47 \\ 0.47 \\ 0.587\,79 \\ 0.693\,15 \end{bmatrix},$$

which we solve to find the coefficients of the polynomial. We get

$$P_5(x) = 0.99910735x - 0.48907554x^2 + 0.28249626x^3 - 0.12895295x^4 + .02\,957206x^5.$$

The following figure shows the difference

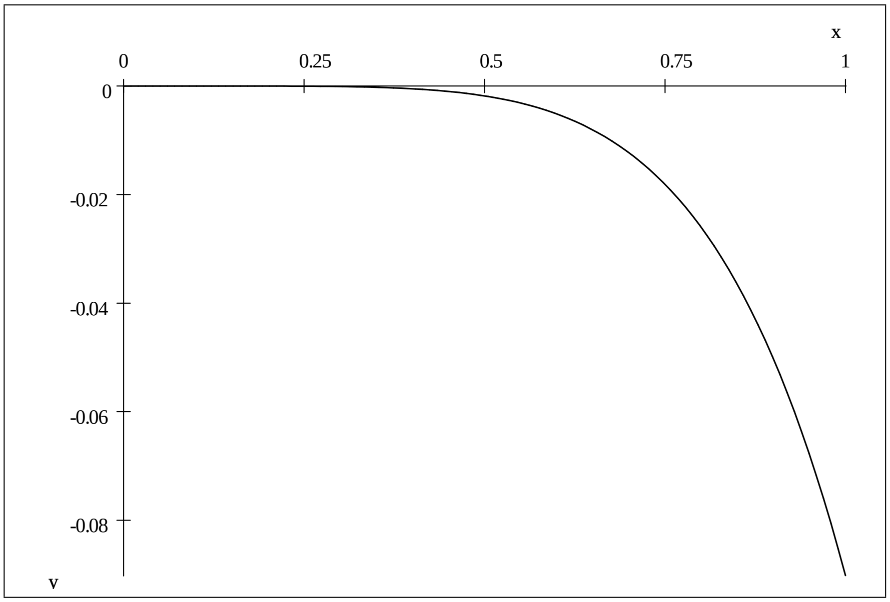
$$\ln(1+x) - P_5(x).$$




Observe that the difference is exactly zero at the points $0, 0.2, 0.4, 0.6, 0.8, 1$ and is very small elsewhere. On the other hand, if we use the Taylor polynomial

$$P_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

we get the following error figure




which shows good approximation twoards 0 but worse approximation towards 1.

## 1.1 Interpolation and Extrapolation

If $P_n(x)$ is the interpolating polynomial for a function $f$ at the points $x_0, x_1, \ldots, x_n$, and $x \in [a, b]$ then we have one of two possibilities:

1. $x \in (x_k, x_{k+1})$ for some $k$, in which case we call $P_n(x)$ the interpolated value of $f(x)$, or
2. $x < x_0$ or $x > x_n$, in which case we call $P_n(x)$ the extrapolated value of $f(x)$.

If $P_n(x)$ is an interpolating polynomial for $f$ at the points $x_0, x_1, \ldots, x_n$, then (intuitively)

$$P_n'(x) = na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + a_1$$

approximates $f'(x)$ and

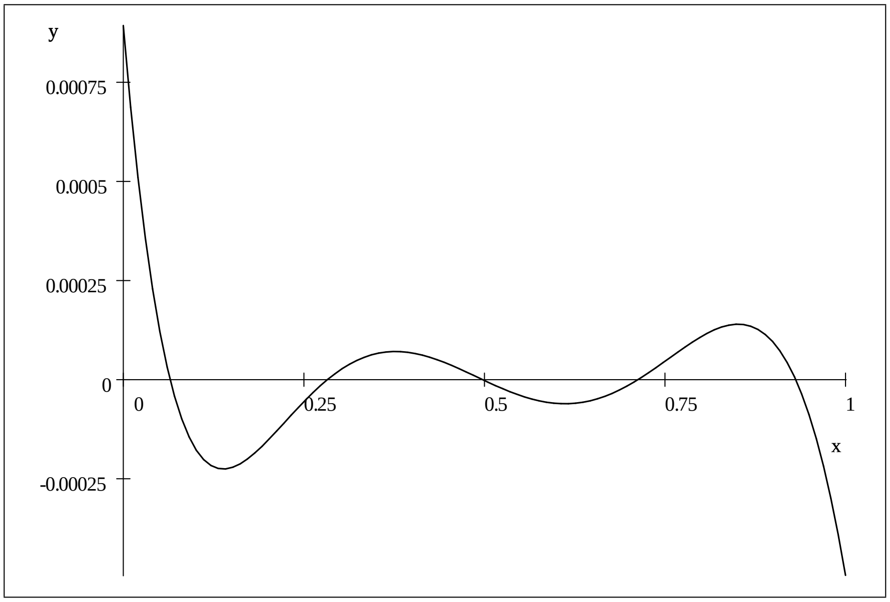
$$\int P_n(x) = \frac{1}{n+1} a_n x^{n+1} + \frac{1}{n} a_{n-1} x^n + \ldots + \frac{1}{2} a_1 x^2 + a_0 x + C$$

approximates

$$\int f(x) + C.$$

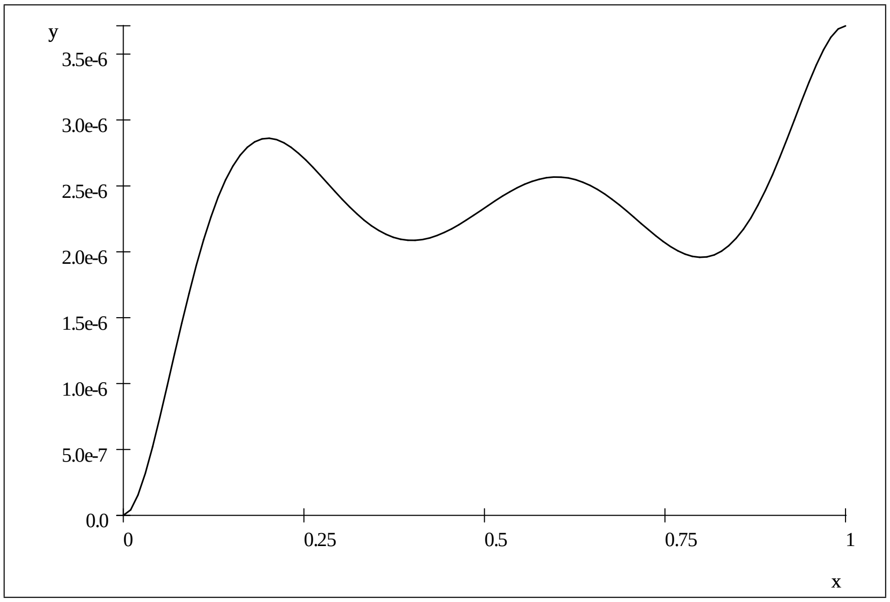
**Example** In the previous example, $P_5'(x) = 0.999\,11 - 0.978\,15x + 0.847\,49x^2 - 0.515\,81x^3 + 0.147\,86x^4$. The following figure shows error

$$(\ln(1+x))' - P_5'(x) = \frac{1}{1+x} - P_5'(x).$$




Also, $\int P_5(x) = 0.499\,55x^2 - 0.163\,03x^3 + 7.062\,4 \times 10^{-2}x^4 - 2.579\,1 \times 10^{-2}x^5 + 4.928\,7 \times 10^{-3}x^6$ approximates the function $\int \ln(1+x)$. The following figure shows

$$\int \ln(1+x) - \int P_5(x) = (x+1)\ln(x+1) - x - \int P_5(x)$$




Horner's method can easily be modified to compute $P_n'(x)$ and $\int P_n(x)$ as follows:

**Horner's Method for the Derivative**

1. *input the coefficients of the polynomial* $A = [a_n, a_{n-1}, \ldots, a_1, a_0]$ and the value $x$
2. *initialise* $p = na_n$
3. *Loop for* $k = 1 : n - 1$

   $p = p * x + (k-1)\, a_{k-1}$

   *end loop*

**Horner's Method for the Integral**

1. *input the coefficients of the polynomial* $A = [a_n, a_{n-1}, \ldots, a_1, a_0]$ and the value $x$
2. *initialise* $p = a_n / (n+1)$
3. *Loop for* $k = 1 : n$

   $p = p * x + a_{k-1}/k$

   *end loop*