1 Taylot Polynomials and the computation of functions

The basis of using Taylor polynomials to approximate values of functions is the following theorem

Theorem 1 *Suppose* $f \in C^{n+1}[a,b]$ *and* $x_0, x \in [a,b]$. *Then*

$$f(x) = P_n(x) + E_n(x)$$

where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

and

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

and where ξ *is in the open interval joining* x_0 *and* x.

Remarks

1. $P_n(x)$ is called the n^{th} Taylor polynomial. Its degree is, in general less than or equal to n. This is because the coefficient of the highest power of x may happen to be zero. For example the 5^{th} Taylor polynomial for $f(x) = \cos x$ at $x_0 = 0$ is

$$\begin{aligned} P_5(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\ (&= P_4(x)). \end{aligned}$$

2. The remainder $E_n(x)$ is used in two ways: to estimate the accuracy of approximation if $P_n(x)$ is used to approximate f and/or to determine the polynomial to be used to achieve a predetermined accuracy.

Example Estimate the error in approximating e by taking $P_9(x)$ with $x_0 = 0$.

Solution: From the remainder formula,

$$E_9(x) = \frac{e^{\xi}}{10!}(x - 0)^{11},$$

where ξ is in the interval joining x to 0. Therefore, with $x = 1$ and since in this case $\xi \in (0,1)$

$$\begin{aligned} E_9(1) &= \frac{e^{\xi}}{10!}(1 - 0)^{11} = \frac{e^{\xi}}{10!} \\ &< \frac{e}{10!} < \frac{3}{10!} = 8.2672 \times 10^{-7} \\ &< 5 \times 10^{-6}. \end{aligned}$$

This also means that $P_9(1)$ approximates e with 6 decimal place accuracy.

Example Find n such that $P_n(x)$ with $x_0 = 0$ approximates $\cos 4$ to 13 decimal places.

Solution: We need to find n such that

$$|E_n(4)| < 5 \times 10^{-13}.$$

But

$$\begin{aligned} |E_n(4)| &= \frac{\left\{ \begin{array}{c} |\cos \xi|, \ n \text{ even} \\ |\sin \xi|, \ n \text{ odd} \end{array} \right.}{(n+1)!} (4-0)^n \\ &\leq \frac{4^n}{(n+1)!}. \end{aligned}$$

Therefore, we need to find n such that

$$\frac{4^n}{(n+1)!} < 5 \times 10^{-13}.$$

By trial and error we find that $\frac{4^{25}}{26!} = 2.7918 \times 10^{-12}$ and $\frac{4^{26}}{27!} = 4.1360 \times 10^{-13}$. Therefore, we take $n = 26$. Therefore, we take

$$P_{26}(4) = 1 - \frac{4^2}{2!} + \frac{4^4}{4!} - \frac{4^6}{6!} + \ldots - \frac{4^{26}}{26!}.$$

1.1 Methods of Polynomial Evaluation

Suppose we want to compute the value of the polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

at some value x. The direct evaluation of this polynomial requires $\frac{(n+1)(n+2)}{2}$ multiplications and n additions and is prone to calncellation error (consider what happens when you evaluate the polynomial $x^2 - 821x$ at $x = 819$ on a 3 digit mantissa). Horner's method (also known as the synthetic division method) evaluates $P_n(x)$ by writing it first in the form

$$P_n(x) = x(\ldots x(x(a_n x + a_{n-1}) + a_{n-2}) + \ldots + a_1) + a_0$$

and then removing the paranthesis from the innermost to the outermost. This method requires only $(n+1)$ multiplications and the same number of additions and avoid cancellation by doing addition before multiplication.

Example To evaluate $P(x) = 3x^4 - 4x^3 + 2x - 1$ at $x = 2$, we write

$$P(x) = x(x(x(3x-4)+0)+2) - 1.$$

Then

$$\begin{aligned} P(2) &= 2(2(2(3*2-4)+0)+2)-1 \\ &= 2(2(2(2)+0)+2)-1 \\ &= 2(2(4)+2)-1 \\ &= 2(10)-1 \\ &= 19, \end{aligned}$$

which can also be verified by direct subsititution.

Algorithm for Horner's Method

1. *input the coefficients of the polynomial* $A = [a_n, a_{n-1}, \ldots, a_1, a_0]$ and the value x
2. *initialise* $p = a_n$ $(= A(1))$
3. *Loop for* $k = 1 : n$

   $p = p * x + a_{k-1}$ $(= A(k+1))$

   end loop