Interpolation by Spline Functions

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One disadvantage of polynomial interpolation is that the more points we have to interpolate the higher the degree of the interpolating polynomial. An interpolating polynomial of degree n can have $n-1$ maxima and minima. This means that the polynomial tends to oscillate between the interpolation points. Another problem wiht high degree polynomials is teh computational time it requires to evaluate. To avoid these problems, we could interpolate with functions which are piecewise polynomials between the interpolating points. This allows us to keep the degree of each polynomial piece at a low value. The typicale polynomial pieces used are of degree one (piecewise linear interpolation), degree 2 (piecewise quadratic interpolation) or degree 3 (piecewise cubic interpolation). In rare occasions degrees higher than 4 are used. Thus, the question we want to answer in this section is: Given a set of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, where we assume that $a = x_0 < x_1 < \ldots < x_n = b$, how can we find an interpolating function $S(x)$ (that is $S(x_k) = y_k$) such that its restriction $S_k(x)$ to the interval $[x_k, x_{k+1}]$, $k = 0, 1, \ldots, n-1$ is a polynomial of a fixed degree r? The function $S(x)$ that has this property is called a spline functions and the polynomial pieces are $S_k(x)$ called splines.

In practical applications to spline interpolation, it is more interesting to use piecewise linear splines ($r = 1$) and piecewise cubic splines ($r = 3$). For this reason, our discussion will be limited to these two types of spline functions.

1 Piecewise Linear Spline interpolation

This case is fairly straight forward to handle. Since $S_k(x)$ is a line segment that passes through the points $(x_k, y_k), (x_{k+1}, y_{k+1})$, we can use Lagrange interpolation to obtain the formula

$$\begin{aligned} S_k(x) &= y_k \frac{x - x_k}{x_{k+1} - x_k} + y_{k+1} \frac{x_{k+1} - x}{x_{k+1} - x_k} \\ &= y_k + d_k (x - x_k), \end{aligned}$$

where

$$d_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}.$$

Therefore, the piecewise linear spline is given by

$$S(x) = y_k + d_k (x - x_k), \text{ if } x \in [x_k, x_{k+1}], \; k = 0, 1, \ldots, n-1.$$

1.1 Calculation with Piecewise Linear Splines

Given a value $x \in [a, b]$, to calculate $S(x)$, we calculate $x - x_1, x - x_2, \ldots, x - x_{k+1}$ and stop when $k + 1$ is the first time the quantity $x - x_{k+1}$ becomes negative. This means that $x \in [x_k, x_{k+1}]$ and, therefore, $S(x) = S_k(x)$.

2 Piecewise Cubic Spline interpolation

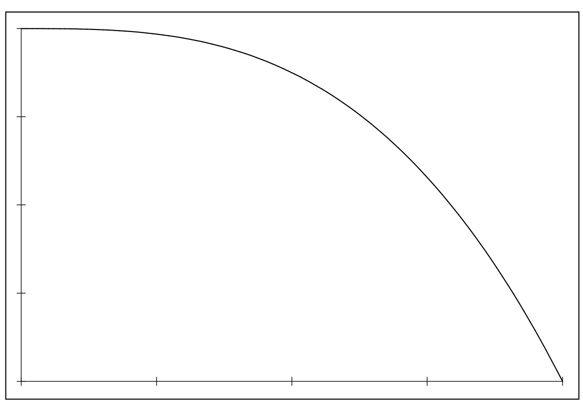
We want to find an interpolating spline function $S(x)$ that has continuous first and second derivatives. That is we want $S(x) \in C^2[a, b]$. This means that the splines $S_k(x)$ will have to be of degree at least 3. This is why this spline function is called a cubic spline. In order to construct $S(x)$, the following conditions must be satisfied.

1. $S(x_k) = S_k(x_k) = y_k,\ k = 0, 1, \ldots, n-1,$
2. $S(x_{k+1}) = S_k(x_{k+1}) = y_{k+1},\ k = 0, 1, \ldots, n-1,$
3. $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1}),\ k = 0, 1, \ldots, n-2,$
4. $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1}),\ k = 0, 1, \ldots, n-2.$

For the ease of construction let's introduce the following four basic splines defined on the interval $[x_k, x_{k+1}]$. Lets denots $x_{k+1} - x_k$ by h_k.

1. The spline $B_{k,0}(x)$ wihg definition

$$B_{k,0}(x) = 1 - \frac{(x - x_k)^3}{h_k^3}.$$



This spline has the following properties:

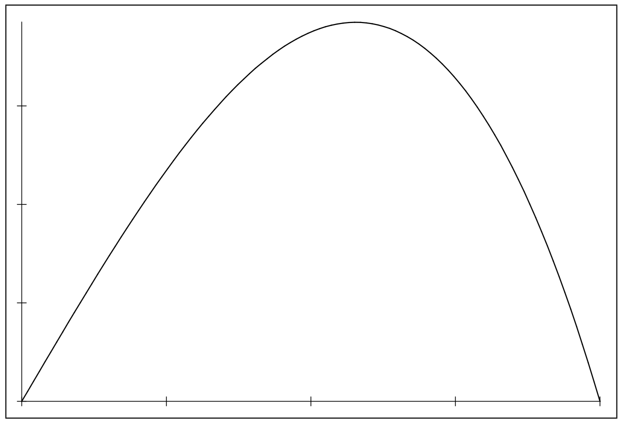
$$B_{k,0}(x_k) = 1,\ B'_{k,0}(x_k) = 0,\ B''_{k,0}(x_k) = 0,\ B_{k,0}(x_{k+1}) = 0.$$

Obswerve also that

$$B'_{k,0}(x) = -3\frac{(x - x_k)^2}{h_k^3},\ B''_{k,0}(x) = -6\frac{(x - x_k)}{h_k^3}$$

2. The spline $B_{k,1}(x)$ wihg definition

$$B_{k,1}(x) = -\frac{(x - x_k)^3}{h_k^2} + (x - x_k)$$



This spline has the following properties:

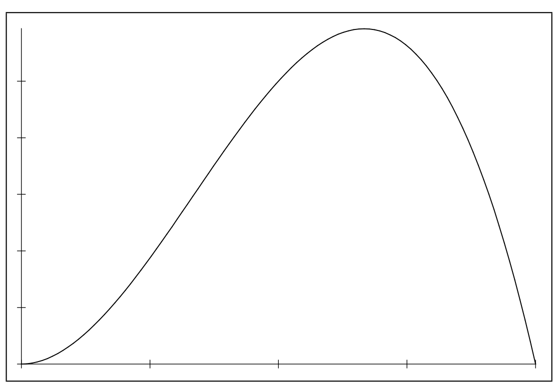
$$B_{k,1}(x_k) = 0,\ B'_{k,1}(x_k) = 1,\ B''_{k,1}(x_k) = 0,\ B_{k,1}(x_{k+1}) = 0.$$

Obswerve also that

$$B'_{k,1}(x) = -3\frac{(x - x_k)^2}{h_k^2} + 1,\ B''_{k,1}(x) = -6\frac{(x - x_k)}{h_k^2}$$

3. The spline $B_{k,2}(x)$ wihg definition

$$B_{k,2}(x) = -\frac{(x - x_k)^3}{2h_k} + \frac{1}{2}(x - x_k)^2$$



This spline has the following properties:

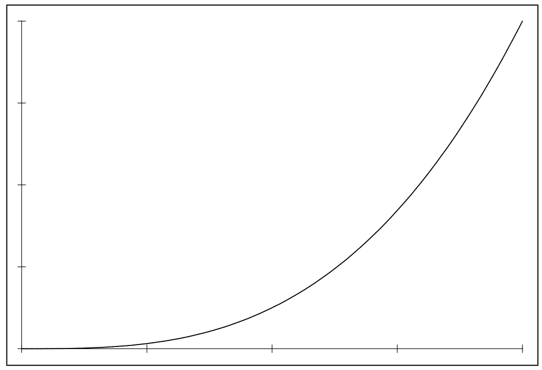
$$B_{k,2}(x_k) = 0,\ B'_{k,2}(x_k) = 0,\ B''_{k,2}(x_k) = 1,\ B_{k,2}(x_{k+1}) = 0.$$

Obswerve also that

$$B'_{k,2}(x) = -3\frac{(x - x_k)^2}{2h_k} + (x - x_k),\ B''_{k,2}(x) = -3\frac{(x - x_k)}{h_k} + 1.$$

4. The spline $B_{k,3}(x)$ wihg definition

$$B_{k,3}(x) = \frac{(x - x_k)^3}{h_k^3}$$



This spline has the following properties:

$$B_{k,3}(x_k) = 0,\ B'_{k,3}(x_k) = 0,\ B''_{k,3}(x_k) = 0,\ B_{k,3}(x_{k+1}) = 1.$$

Obswerve also that

$$B'_{k,3}(x) = 3\frac{(x - x_k)^2}{h_k^3},\ B''_{k,3}(x) = 6\frac{(x - x_k)}{h_k^3}.$$

The reason for introducing these basic splines is tha we can now write the spline $S_k(x)$ as

$$S_k(x) = s_{k,0}B_{k,0} + s_{k,1}B_{k,1} + s_{k,2}B_{k,2} + s_{k,3}B_{k,3}$$

and we now need to find $s_{k,0}, s_{k,1}, s_{k,2}, s_{k,3}$ such that the 4 properties listed above of $S_k(x)$ are satisfied. The condition

$$S_k(x_k) = y_k$$

gives

$$y_k = s_{k,0},\ k = 0, 1, \ldots, n - 1.$$

The condition

$$S_k(x_{k+1}) = y_{k+1}$$

gives

$$y_{k+1} = s_{k,3}\ k = 0, 1, \ldots, n - 1.$$

The condition

$$S'_k(x_{k+1}) = S'_{k+1}(x_{k+1}),\ k = 0, 1, \ldots, n - 2$$

gives

$$s_{k+1,1} = -\frac{3}{h_k}y_k - 2s_{k,1} - \frac{h_k}{2}s_{k,2} + \frac{3}{h_k}y_{k+1},\ k = 0, 1, \ldots, n - 2. \qquad (1)$$

Finally, the condition

$$S_k''(x_{k+1}) = S_{k+1}''(x_{k+1}), \ k = 0, 1, \ldots, n-2$$

gives

$$s_{k+1,2} = -\frac{6}{h_k^2} y_k - \frac{6}{h_k} s_{k,1} - 2s_{k,2} + \frac{6}{h_k^2} y_{k+1}, \ k = 0, 1, \ldots, n-2. \tag{2}$$

At this point we notice that we have n splines $S_0(x), S_1(x), \ldots, S_{n-1}(x)$, each of which depends on two, so far undetermined parameters: $s_{k,1}, s_{k,2}$ $k = 0, 1, \ldots, n-2$. Therefore, we have $2n$ unknowns, and $2(n-1)$ equations, namely (1) equations (2) to determine them. This means that we have two free variables in this system. We need to more pieces of information to uniquely determine all the $2n$ parameters. In practice these extra conditions are set at the boundaries $x_0 = a$ and/or $x_n = b$. In order to see what kind of extra conditions can be imposed, let's write equations (1), (2) in matrix form. For this purpose set

$$\begin{aligned} W_k &= \begin{bmatrix} s_{k,1} \\ s_{k,2} \end{bmatrix}, Y_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}, \\ A_k &= \begin{bmatrix} -\frac{3}{h_k} & \frac{3}{h_k} \\ -\frac{6}{h_k^2} & \frac{6}{h_k^2} \end{bmatrix}, B_k = \begin{bmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{bmatrix}. \end{aligned}$$

Then equations (1), (2) can be written as

$$W_{k+1} = A_k Y_k + B_k W_k, \ k = 0, 1, \ldots, n-2.$$

Starting with $k = 0$ we have

$$W_1 = A_0 Y_0 + B_0 W_0 = Z_0 + D_0 W_0,$$

where

$$Z_0 = A_0 Y_0, D_0 = B_0$$

For $k = 1$, we have

$$\begin{aligned} W_2 &= A_1 Y_1 + B_1 W_1 \\ &= A_1 Y_1 + B_1 (Z_0 + D_0 W_0) \\ &= A_1 Y_1 + B_1 Z_0 + B_1 D_0 W_0 \\ &= Z_1 + D_1 W_0, \end{aligned}$$

where

$$Z_1 = A_1 Y_1 + B_1 Z_0, D_1 = B_1 D_0.$$

Continuing by induction we get

$$W_k = Z_{k-1} + D_{k-1} W_0, \tag{3}$$

where

$$Z_k = A_k Y_k + B_k Z_{k-1}, D_k = B_k D_{k-1}$$

Therefore, we have

$$W_n = Z_{n-1} + D_{n-1} W_0$$

Observe that

$$\begin{aligned} W_n &= \begin{bmatrix} S'_{n-1}(b) \\ S''_{n-1}(b) \end{bmatrix} = \begin{bmatrix} S'(b) \\ S''(b) \end{bmatrix}, \\ W_0 &= \begin{bmatrix} s_{0,1} \\ s_{0,2} \end{bmatrix} = \begin{bmatrix} S'_0(a) \\ S''_0(a) \end{bmatrix} = \begin{bmatrix} S'(a) \\ S''(a) \end{bmatrix}. \end{aligned}$$

Therefore, we have the relations between the end points:

$$\begin{bmatrix} S'(b) \\ S''(b) \end{bmatrix} = Z_{n-1} + D_{n-1} \begin{bmatrix} S'(a) \\ S''(a) \end{bmatrix}. \tag{4}$$

The most important feature about this system is that the matrix

$$D_{n-1} = B_{n-1} B_{n-2} \cdots B_0.$$

is nonsingular because each B_k has determinant 1. Furthermore, $D_{n-1} = (-1)^n |B_{n-1}| |B_{n-2}| \cdots |B_0|$, where $|B_k|$ means the absolute value of elements in B_k. Therefore, all entries of D_{n-1} are nonzero. As a consequence of all this, specifying any two of the four quantities $S'_0(a), S''_0(a), S'_{n-1}(b), S''_{n-1}(b)$ uniquely determines the other two. Once W_0 is completely determined, equation (3) completely determines W_k and, thus, completely determine $S_k(x)$. Practical sepcifications of the above 4 values are:

1. *Clamped spline:* $S'(a) = d_0, S'(b) = d_n$ are specified.

   Writing equation (3) out in more detail gives

   $$\begin{aligned} \begin{bmatrix} S'(b) \\ S''(b) \end{bmatrix} &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} S'(a) \\ S''(a) \end{bmatrix} \\ \begin{bmatrix} d_n \\ S''(b) \end{bmatrix} &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} d_0 \\ S''(a) \end{bmatrix}. \end{aligned}$$

   Thus $d_n = z_1 + d_{11} d_0 + d_{12} S''(a)$, which determines $S''(a)$ and completes the determination of W_0.

2. *Endpoint Curvature Adjusted spline:* $S''(a) = c_0, S''(b) = c_n$ are specified. In this case $c_n = z_2 + d_{21} S'(a) + d_{22} c_0$which determines $S'(a)$, and consequently, completes the determination of W_0.

3. *Natural Spline:* $S''(a) = 0, S''(b) = 0$. This is a special case of the above one.

4. *The extrapolated spline:* Here, we assume that the spline S_1 passes through y_0 and the spline S_{n-2} passes through y_n. Then

$$\begin{aligned} y_0 &= y_1 B_{1,0}(x_0) + s_{1,1} B_{1,1}(x_0) + s_{1,2} B_{1,2}(x_0) + y_2 B_{1,3}(x_0), \\ y_n &= y_{n-2} B_{n-2,0}(x_n) + s_{n-2,1} B_{n-2,1}(x_n) + s_{n-2,2} B_{1,2}(x_n) + y_{n-1} B_{1,3}(x_n). \end{aligned}$$

This system can be rewritten in the form

$$\begin{aligned} RW_1 &= \alpha, \\ SW_{n-1} &= \beta, \end{aligned}$$

where

$$\begin{aligned} R &= [B_{1,1}(x_0)\ \ B_{1,2}(x_0)] = \left[\left(\frac{h_0^3}{h_1^2} - h_0 \right)\ \left(\frac{h_0^3}{2h_1} - \frac{h_0^2}{2} \right) \right], \\ S &= [B_{n-2,1}(x_n)\ \ B_{1,2}(x_n)] = \left[\left(\frac{h_n^3}{h_{n-1}^2} - h_n \right)\ \left(\frac{h_n^3}{2h_{n-1}} - \frac{h_n^2}{2} \right) \right], \\ \alpha &= [B_{1,0}(x_0)\ \ B_{1,3}(x_0)]\, Y_1 = \left[\left(1 + \frac{h_0^3}{h_1^3} \right)\ \frac{h_0^3}{h_1^3} \right] Y_1, \\ \beta &= [B_{n-2,0}(x_n)\ \ B_{1,3}(x_n)]\, Y_{n-2} = \left[\left(1 + \frac{h_n^3}{h_{n-1}^3} \right)\ \frac{h_n^3}{h_{n-1}^3} \right]. \end{aligned}$$

Observing that W_{n-2} and W_1 are related by an equation of the form

$$W_{n-1} = Z_{n-2} + D_{n-2} W_1,$$

we have the system

$$\begin{aligned} RW_1 &= \alpha, \\ SD_{n-2} W_1 &= \beta - SZ_{n-2}. \end{aligned}$$

which can be solved to get W_1.

5. *The parabolic terminated spline:* In this case S_0 and S_{n-1} are parabolic splines (the basic functions for these splines are left as an exercise). Since each one of these splines depends on 3 parameters rather than 4, the number of overall variables matches the number of equation. Applying the continuity conditions for S_0 and S_{n-1}, we end up with a similar situation to the extrapolated splines.

Example Find the cubic spline that interpolates the function $f(x) = x + 2/x$ at the nodes $x_0 = 1/2$, $x_1 = 1$, $x_2 = 3/2$, $x_3 = 2$, assuming a clamped spline configuration, with $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$. In this problem we have

$$\begin{aligned}
Y_0 &= \begin{bmatrix} 4.5 \\ 3 \end{bmatrix}, Y_1 = \begin{bmatrix} 3 \\ 17/6 \end{bmatrix}, Y_2 = \begin{bmatrix} 17/6 \\ 3 \end{bmatrix}, \\
A_0 &= \begin{bmatrix} -6 & 6 \\ -24 & 24 \end{bmatrix} = A_1 = A_2, \\
B_0 &= \begin{bmatrix} -2 & -\frac{1}{4} \\ -12 & -2 \end{bmatrix} = B_1 = B_2, \\
Z_0 &= A_0 Y_0 = \begin{bmatrix} -9 \\ -36 \end{bmatrix}, Z_1 = A_1 Y_1 + B_1 Z_0 = \begin{bmatrix} 26 \\ 176 \end{bmatrix}, \\
Z_2 &= A_2 Y_2 + B_2 Z_1 = \begin{bmatrix} -95 \\ -660 \end{bmatrix} \\
D_1 &= B_1 D_0 = \begin{bmatrix} 7 & 1 \\ 48 & 7 \end{bmatrix}, D_2 = B_2 D_1 = \begin{bmatrix} -26 & -\frac{15}{4} \\ -180 & -26 \end{bmatrix}.
\end{aligned}$$

The equation

$$W_3 = Z_2 + D_2 W_0$$

takes the form

$$\begin{bmatrix} \frac{1}{2} \\ S''(2) \end{bmatrix} = \begin{bmatrix} -95 \\ -660 \end{bmatrix} + \begin{bmatrix} -26 & -\frac{15}{4} \\ -180 & -26 \end{bmatrix} \begin{bmatrix} -7 \\ S''(1/2) \end{bmatrix}.$$

Solving for $S''(1/2)$, we get

$$S''(1/2) = 23.0667.$$

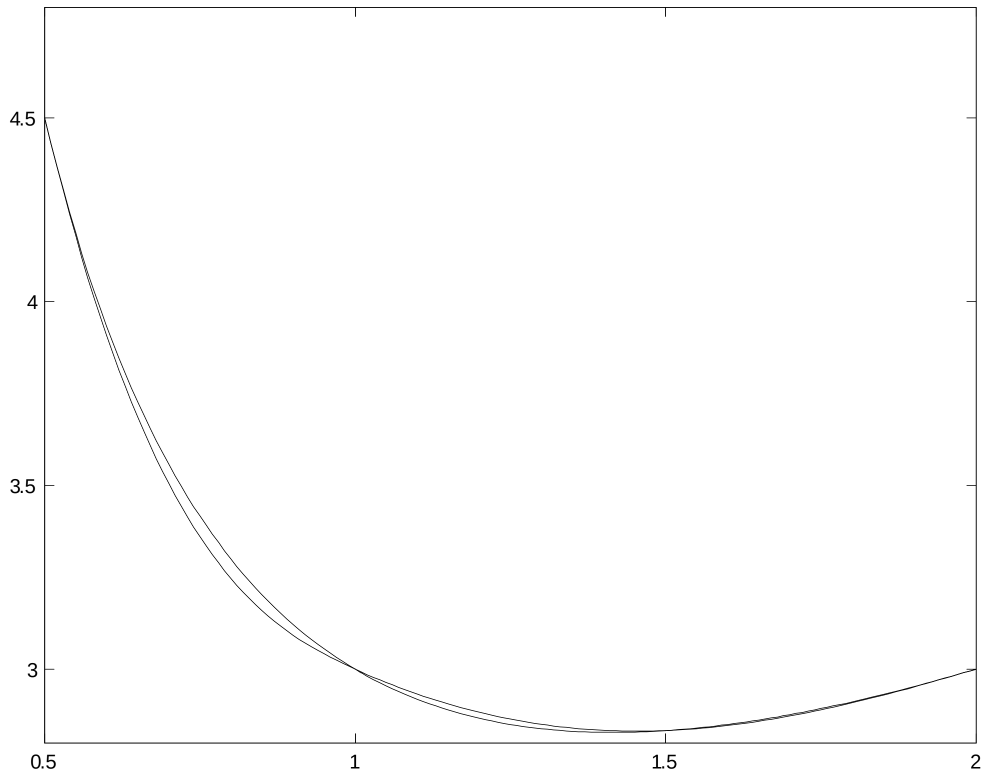
Thus

$$W_0 = \begin{bmatrix} -7 \\ 23.0667 \end{bmatrix}, W_1 = \begin{bmatrix} -.7667 \\ 1.8667 \end{bmatrix}, W_2 = \begin{bmatrix} .0667 \\ 1.4667 \end{bmatrix}.$$

the coefficients of the spline S_0 are $[4.5,\ -7.,\ 23.0667,\ 3\ .]$, of S_1 are $[3.,\ -0.7667,\ 1.8667,\ 2.8333]$ and of S_2 are $[2.8333,\ 0.0667,\ 1.4667,\ 3.]$. Therefore,

$$\begin{aligned}
S_0(x) &= 4.5 B_{0,0}(x) - 7 B_{0,1}(x) + 23.0667 B_{0,2}(x) + 3 B_{0,3}(x), \\
S_1(x) &= 3 B_{1,0}(x) - 0.7667 B_{1,1}(x) + 1.8667 B_{1,2}(x) + 2.8333 B_{1,3}(x) \\
S_2(x) &= 2.8333 B_{2,0}(x) + 0.0667 B_{2,1}(x) + 1.4667 B_{2,2}(x) + 3 B_{2,3}(x)
\end{aligned}$$

The following figure shows the spline function and the original function drawn on the same graph.


4.5
4
3.5
3
0.5
1
1.5
2