Newton Polynomial Interpolation

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1 Newton Polynomials

Newton polynomials provide an alternative method for interpolating a function to Lagrange polynomials. Suppose $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct nodes and $a_0, a_1, \ldots, a_n$ are real numbers. Newton polynomials are defined by the recursive formula

$$
\begin{aligned}
P_0(x) &= a_0, \\
P_k(x) &= P_{k-1}(x) + a_k (x - x_0)(x - x_1) \ldots (x - x_{k-1}), \ k = 1, 2, \ldots, n.
\end{aligned}
$$

The points $x_0, x_1, \ldots, x_{k-1}$ are called the centers of the polynomial $P_k(x)$. Thus $P_1(x) = P_0(x) + a_1(x - x_0)$ and $P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$ and so on. This recursive definition of Newton polynomials has advantages (over the lagrange polynomials). The computation of the kth polynomial makes use of the already computed earlier ones. When more nodes are added we only need to compute the polynomials corresponding to the newer nodes. Furthermore, the computation of the newer polynomials builds on the ones already computed. In the case of the Lagrange interpolation, all polynomials have to be computed seperately.

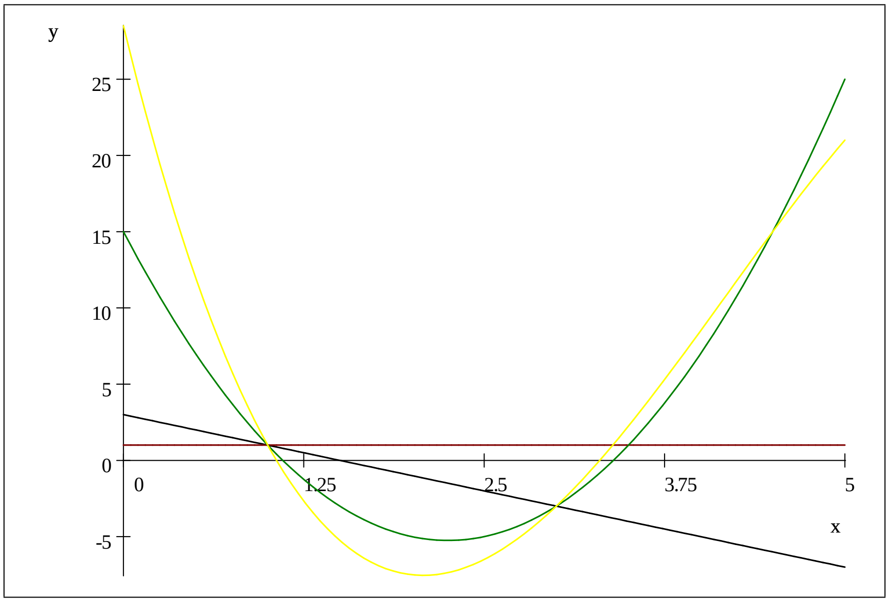
Example Given the data

k	0	1	2	3
x_k	1	3	4	4.5
a_k	1	-2	4	-1

,

the corresponding Newton polynomials are

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= 1 - 2(x - 1) = -2x + 3, \\
P_2(x) &= 3 - 2x + 4(x - 1)(x - 3) = 4x^2 - 18x + 15, \\
P_3(x) &= 4x^2 - 18x + 15 - (x - 1)(x - 3)(x - 4.5) = -x^3 + 12.5x^2 - 39.0x + 28.5
\end{aligned}
$$



Observe that all 4 polynomials have the same value at x_0, P_1, P_2, P_3 have the same value at x_1, P_2, P_3 have the same value at x_2. Furthermore, suppose we want to evaluate the polynomials at $x = 2.5$, then

$$\begin{aligned}
P_0(2.5) &= 1, \\
P_1(2.5) &= P_0(2.5) - 2(2.5-1) = 1 - 2(2.5-1) = -2.0, \\
P_2(2.5) &= P_1(2.5) + 4(2.5-1)(2.5-3) = -2.0 + 4(2.5-1)(2.5-3) = -5.0, \\
P_3(2.5) &= P_2(2.5) - (2.5-1)(2.5-3)(2.5-4.5) = -5.0 - (2.5-1)(2.5-3)(2.5-4.5) = -6.5.
\end{aligned}$$

2 Interpolation by Newton Polynomials

We now want to construct the Newton polynomial $P_N(x)$ that will interpolate a function $f(x)$ at the distinct nodes $x_0, x_1, \ldots, x_n$. We begin by introducing the following notation.

Definition 1 *The Newton Divided Differences for a function $f(x)$ are defined by the recursive relation:*

$$\begin{aligned}
f[x_k] &= f(x_k), \\
f[x_{k-j}, x_{k-j+1}, \ldots, x_k] &= \frac{f[x_{k-j+1}, \ldots, x_k] - f[x_{k-j}, \ldots, x_{k-1}]}{x_k - x_{k-j}}.
\end{aligned}$$

For example,

$$\begin{aligned}
f[x_0] &= f(x_0), \\
f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \\
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2]\, f[x_0, x_1]}{x_2 - x_0}, \\
f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}
\end{aligned}$$

and so on.

With this notation we proceed to find the Newton interpolating polynomial as follows. We set

$$\begin{aligned} P_0(x) &= f(x_0) = f[x_0], \\ P_1(x) &= P_0(x) + a_1(x - x_0) \end{aligned}$$

so that

$$P_0(x_0) = f(x_0).$$

In other words, $P_0(x)$ interpolates $f(x)$ at x_0. Next, we set

$$P_1(x) = P_0(x) + a_1(x - x_0)$$

and determine a_1 such that

$$P_1(x_1) = f(x_1).$$

Then

$$\begin{aligned} f(x_1) &= P_0(x_1) + a_1(x_1 - x_0) \\ &= f(x_0) + a_1(x_1 - x_0), \end{aligned}$$

which gives

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1].$$

Thus

$$P_1(x) = f[x_0] + f[x_0, x_1](x_1 - x_0).$$

Observe also that

$$P_1(x_0) = f[x_0] = f(x_0).$$

Therefore, P_1 interpolates f at x_0, x_1. Next, we set

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$$

and determine a_2 so that $P_2(x_2) = f(x_2)$. Then

$$\begin{aligned} f(x_2) &= P_1(x_2) + a_2(x_2 - x_0)(x_2 - x_1) \\ &= f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1). \end{aligned}$$

Rather than solving this equation directly for a_2, we manipulate it as follows

$$\begin{aligned} f(x_2) - f(x_1) &= f(x_0) - f(x_1) + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ \frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{f(x_0) - f(x_1)}{x_2 - x_1} + f[x_0, x_1]\frac{x_2 - x_0}{x_2 - x_1} + a_2(x_2 - x_0) \\ f[x_1, x_2] &= -f[x_0, x_1]\frac{x_1 - x_0}{(x_2 - x_1)} + f[x_0, x_1]\frac{x_2 - x_0}{x_2 - x_1} + a_2(x_2 - x_0) \\ &= f[x_0, x_1]\left(\frac{x_2 - x_0}{x_2 - x_1} - \frac{x_1 - x_0}{x_2 - x_1}\right) + a_2(x_2 - x_0) \\ &= f[x_0, x_1] + a_2(x_2 - x_0). \end{aligned}$$

Therefore,

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2].$$

Hence

$$P_2(x) = P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

Observe also that $P_2(x_0) = P_1(x_0) = f(x_0)$ and $P_2(x_1) = P_1(x_1) = f_1(x_1)$. Therefore, P_2 interpolates f at x_0, x_1, x_2. Proceeding in this manner, we can show by induction that the polynomial

$$\begin{aligned} P_x(x) &= P_{x-1}(x) + f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\ldots(x - x_{k-1}) \\ &= f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\ldots(x - x_{k-1}) \end{aligned}$$

is the polynomial of degree at most k which interpolates f at the nodes $x_0, x_1, \ldots, x_k$. Therefore, the polynomial

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

is the Newton polynomial of degree at most n which interpolates polynomial f at the nodes $x_0, x_1, \ldots, x_n$.

The actual calculation of these polynomials is organised in a table as in the following example.

Example The Newton polynomial that interpolates $\ln(1 + x)$ at the nodes $0, 0.2, 0.4, 0.6, 0.8, 1$ is calculated in tabluated form as follows

k	x_k	D_0	D_1	D_2	D_3	D_4	D_5
0	x_0	$\boxed{f[x_0]}$					
1	x_1	$f[x_1]$	$\boxed{f[x_0, x_1]}$				
2	x_2	$f[x_2]$	$f[x_1, x_2]$	$\boxed{f[x_0, x_1, x_2]}$			
3	x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$\boxed{f[x_0, x_1, x_2, x_3]}$		
4	x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$\boxed{f[x_0, x_1, x_2, x_3, x_4]}$	
5	x_5	$f[x_5]$	$f[x_4, x_5]$	$f[x_3, x_4, x_5]$	$f[x_2, x_3, x_4, x_5]$	$f[x_1, x_2, x_3, x_4, x_5]$	$\boxed{f[x_0, x_1, x_2, x_3, x_4, x_5]}$

k	x_k	D_0	D_1	D_2	D_3	D_4	D_5
0	0	$\boxed{0}$					
1	0.2	0.182 32	$\boxed{0.911 6}$				
2	0.4	0.336 47	0.770 75	$\boxed{-0.352 13}$			
3	0.6	0.47	0.667 65	−0.257 75	$\boxed{0.157 3}$		
4	0.8	0.587 79	0.588 95	−0.196 75	0.101 67	$\boxed{-6.953 8 \times 10^{-2}}$	
5	1	0.693 15	0.526 8	−0.155 38	0.068 95	−0.040 9	$\boxed{2.863 8 \times 10^{-2}}$

Hence,

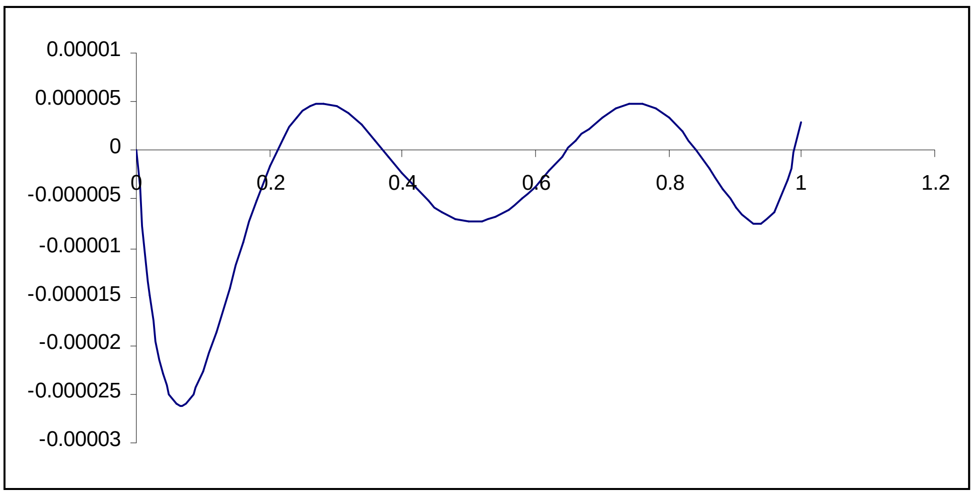
$$\begin{aligned} P_5(x) &= 0.911\,6x - 0.352\,13x(x - .2) + 0.157\,3x(x - .2)(x - .4) \\ &\quad -.06\,953\,8x(x - .2)(x - .4)(x - .6) + .02\,863\,8x(x - .2)(x - 4)(x - .6)(x - .8) \\ &= 0.445\,79x^3 - 0.229\,91x^4 + 2.863\,8 \times 10^{-2}x^5. \end{aligned}$$

The following theorem gives the error formula for interpolation with Newton polynomials. It is exactly the same as that for Lagrange polynomials

Theorem 2 *Assume $f \in C^{n+1}[a, b]$ and $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct nodes. Suppose further that $P_n(x)$ is the Newton polynomial interpolating f at the nodes $x_0, x_1, \ldots, x_n$. Then there is a number ξ in the interval formed by the nodes $x_0, x_1, \ldots, x_n$ such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

Remark 3 *New points can be added to the nodes and in order to obtain the higher degree Newton interpolation polynomial all we need to do is insert the new points at the end of the table and complete the computations of the new table.*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-0.00001
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0.8
1
1.2