Lagrange Interpolation

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1 The Lagrange Basic Polynomials

Suppose f is defined on an interval $[a,b]$ and we have $n+1$ points (called nodes) $x_0, x_1, \ldots, x_n \in [a,b]$ such that $a \leq x_0 \leq x_1 \leq \ldots \leq x_n \leq b$. We want to find the polynomial $P_n(x)$ of degree at most n that interpolates f at the nodes $x_0, x_1, \ldots, x_n$, that is

$$P_n(x_k) = f(x_k), \; k = 0, 1, \ldots, n.$$

We saw in the previous section how this can be done by solving a linear system of equations. The problem with the linear system is that its matrix (known as a Hilbert matrix) is ill conditioned and the solution is very sensitive to representation errors. In this section we will discuss an alternative way of constructing the polynomial $P_n(x)$. We begin by introducing the basic Lagrange polynomials $L_{n,k}, \; k = 0, 1, \ldots, n.$

Suppose we want to construct a polynomial $Q_k(x)$ that satisfies the following conditions:

$$Q_k(x_j) = 0, \; j \neq k, \tag{1}$$

$$Q_k(x_k) = 1. \tag{2}$$

We try a polynomial of the form

$$Q_k(x) = c(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1})(x - x_n).$$

So far $Q_k(x)$ satisfies the conditions (1) but its value at x_k need not be 1. To make its value at x_k equal one we adjust the coefficient c. This results in the equation

$$1 = Q_k(x_k) = c(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1})(x_k - x_n)$$

which determines c as

$$c = \frac{1}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1})(x_k - x_n)}.$$

Thus the polynomial $Q_k(x)$ is

$$Q_k(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1})(x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1})(x_k - x_n)}.$$

To simplify the expression we use the product notation:

$$Q_k(x) = \frac{\prod_{j=0, j\neq k}^{n} (x - x_j)}{\prod_{j=0, j\neq k}^{n} (x_k - x_j)}.$$

Definition 1 *The Lagrange basic polynomial $L_{n,k}(x)$ is defined to be $Q_k(x)$, $k = 0, 1, \ldots, n$.*

From our discussion of how the polynomial $Q_k(x)$ was constructed we see that $L_{n,k}(x)$ has degree n, and it satisfies the conditions

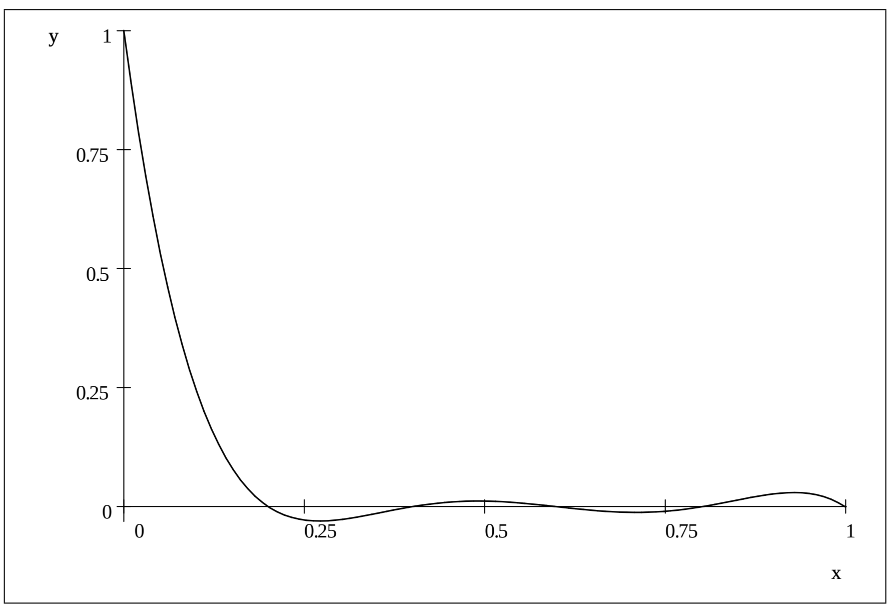
$$L_{n,k}(x_j) = \delta_{kj},$$

where δ_{kj} is the so called Kronecker delta and is defined by

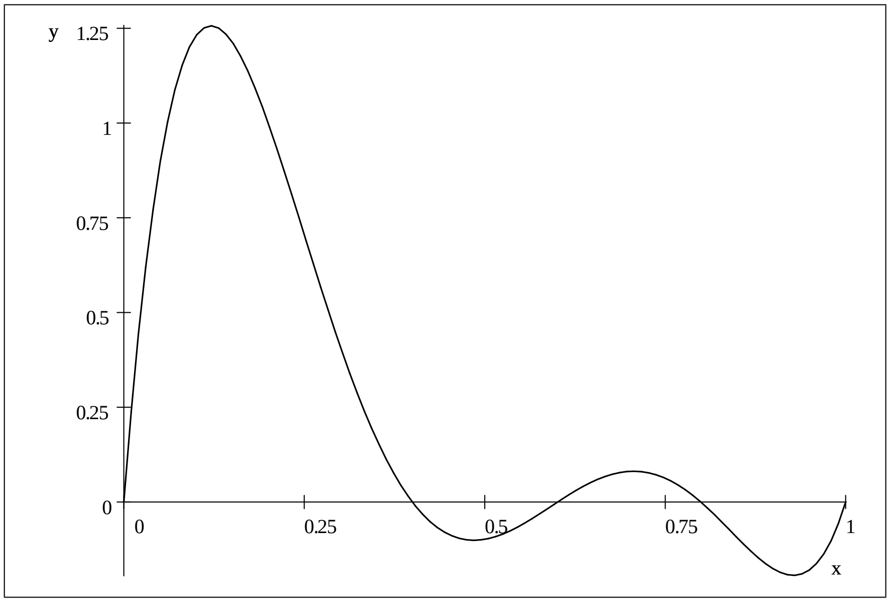
$$\delta_{kj} = \begin{cases} 0, k \neq j \\ 1, \ k = j \end{cases}.$$

Example For the 6 nodes $0, 0.2, 0.4, .6, 0.8, 1 \in [0, 1]$ we have

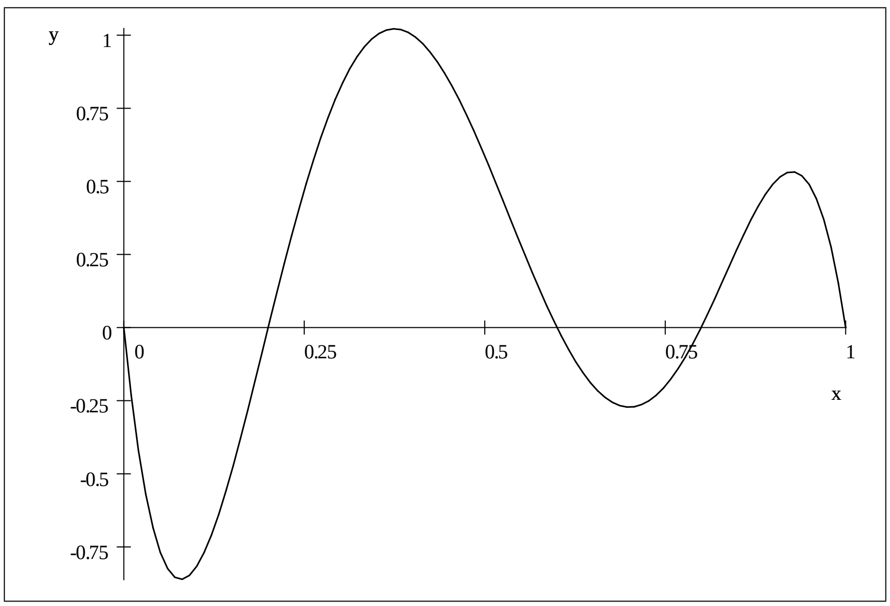
$$\begin{aligned} L_{5,0} &= \frac{(x-0.2)(x-0.4)(x-0.6)(x-0.8)(x-1)}{(0-0.2)(0-0.4)(0-0.6)(0-0.8)(0-1)} \\ &= -26.042x^5 + 78.125x^4 - 88.542x^3 + 46.875x^2 - 11.417x + 1.0, \end{aligned}$$



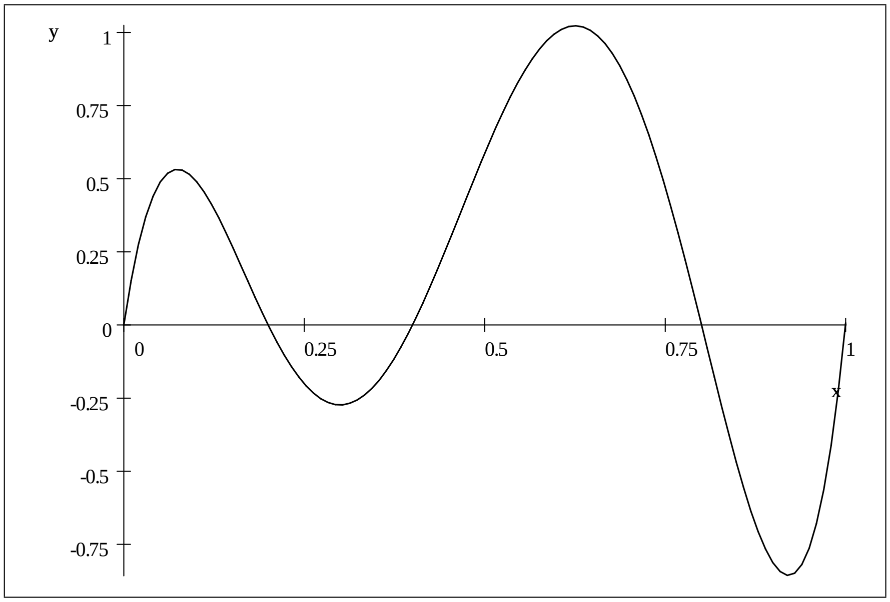
$$\begin{aligned} L_{5,1} &= \frac{x(x-0.4)(x-0.6)(x-0.8)(x-1)}{0.2(0.2-0.4)(0.2-0.6)(0.2-0.8)(0.2-1)} \\ &= 130.21x^5 - 364.58x^4 + 369.79x^3 - 160.42x^2 + 25.0x, \end{aligned}$$



$$L_{5,2} = \frac{x(x-0.2)(x-0.6)(x-0.8)(x-1)}{0.4(0.4-0.2)(0.4-0.6)(0.4-0.8)(0.4-1)}$$
$$= -260.42x^5 + 677.08x^4 - 614.58x^3 + 222.92x^2 - 25.0x,$$



$$L_{5,3} = \frac{x(x-0.2)(x-0.4)(x-0.8)(x-1)}{0.6(0.6-0.2)(0.6-0.4)(0.6-0.8)(0.6-1)}$$
$$= 260.42x^5 - 625.0x^4 + 510.42x^3 - 162.5x^2 + 16.667x,$$



$$L_{5,4} = \frac{x(x-0.2)(x-0.4)(x-0.6)(x-1)}{0.8(0.8-0.2)(0.8-0.4)(0.8-0.6)(0.8-1)}$$
$$= -130.21x^5 + 286.46x^4 - 213.54x^3 + 63.542x^2 - 6.25x,$$

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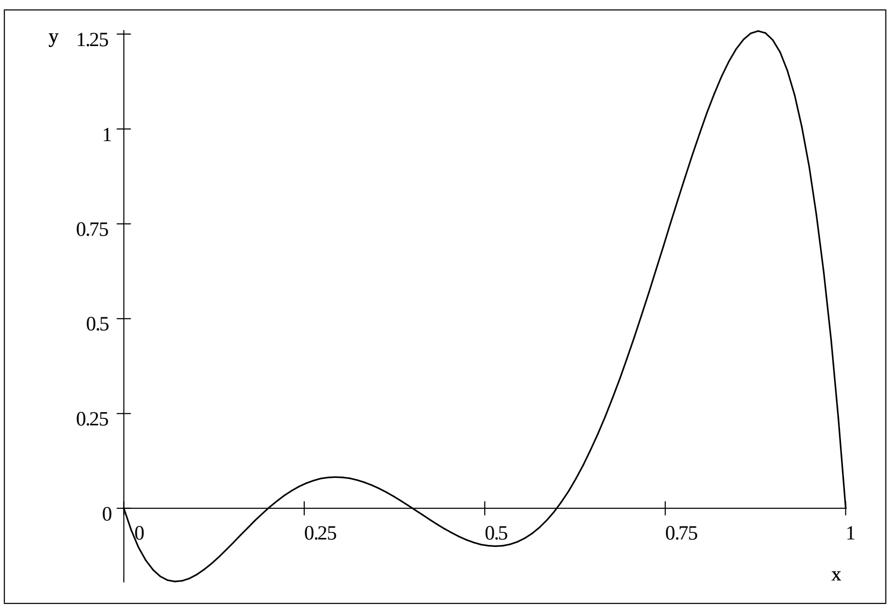
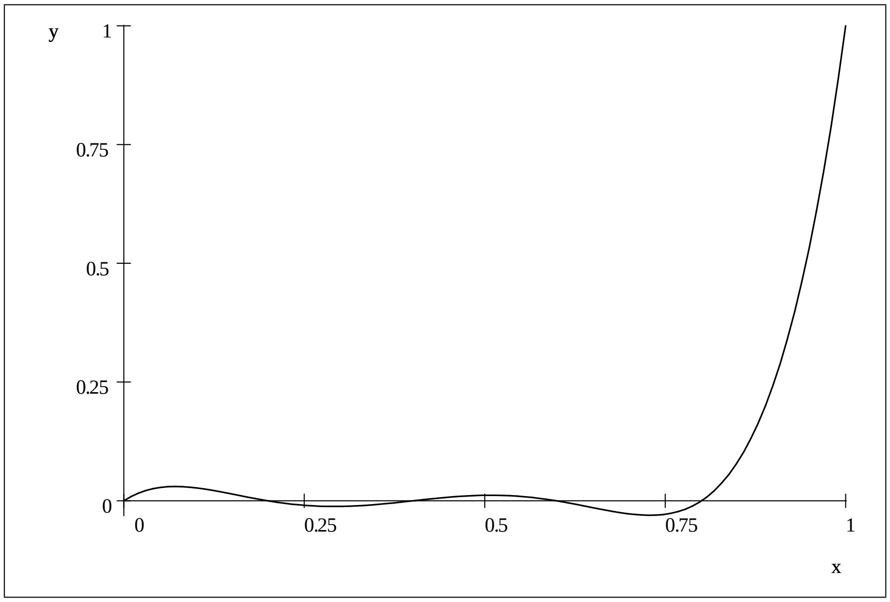
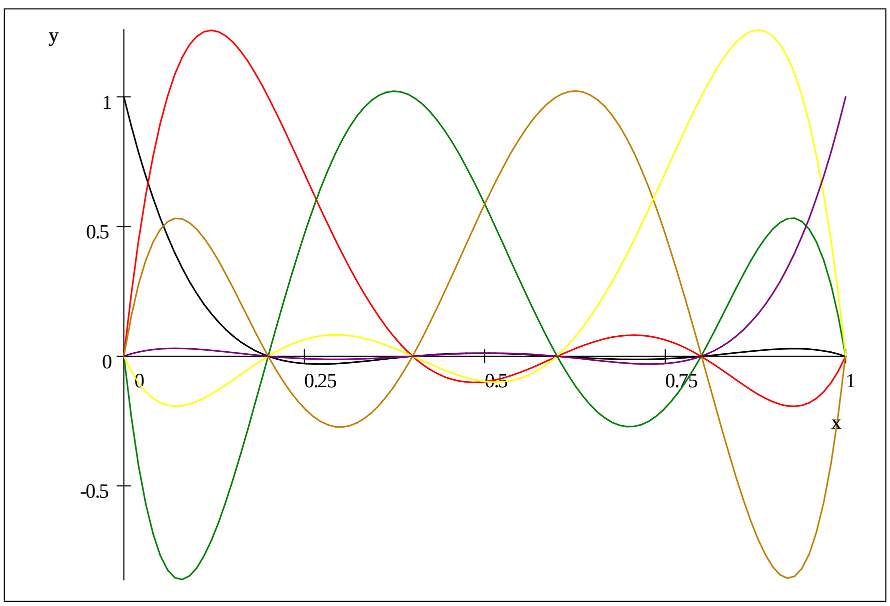


$$L_{5,5} = \frac{x(x-0.2)(x-0.4)(x-0.6)(x-0.8)}{1(1-0.2)(1-0.4)(1-0.6)(1-0.8)}$$
$$= 26.042x^5 - 52.083x^4 + 36.458x^3 - 10.417x^2 + x.$$



The following figure shows all 6 polynomials plotted together.



2 Lagrange Interpolating Polynomial

The polynomial $P_n(x)$ of degree at most n which interpolates f at the nodes $x_0, x_1, \ldots, x_n$ can be defined now with th help of the Lagrange basic polynomials as

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x). \tag{3}$$

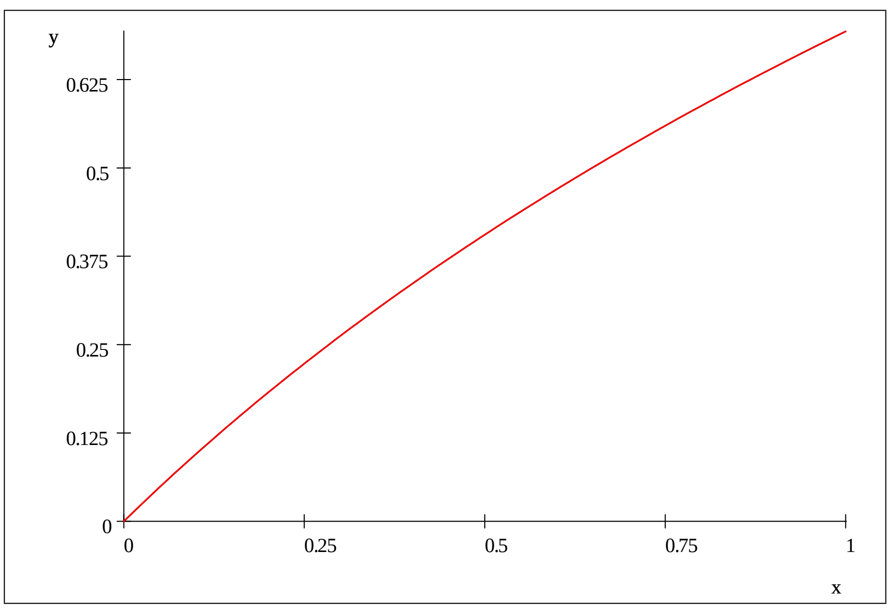
Obseve that

$$\begin{aligned} P_n(x_j) &= \sum_{k=0}^{n} f(x_k) L_{n,k}(x_j) \\ &= \sum_{k=0}^{n} f(x_k)\delta_{kj} = f(x_j), \ \ j = 0, 1, \ldots, n. \end{aligned}$$

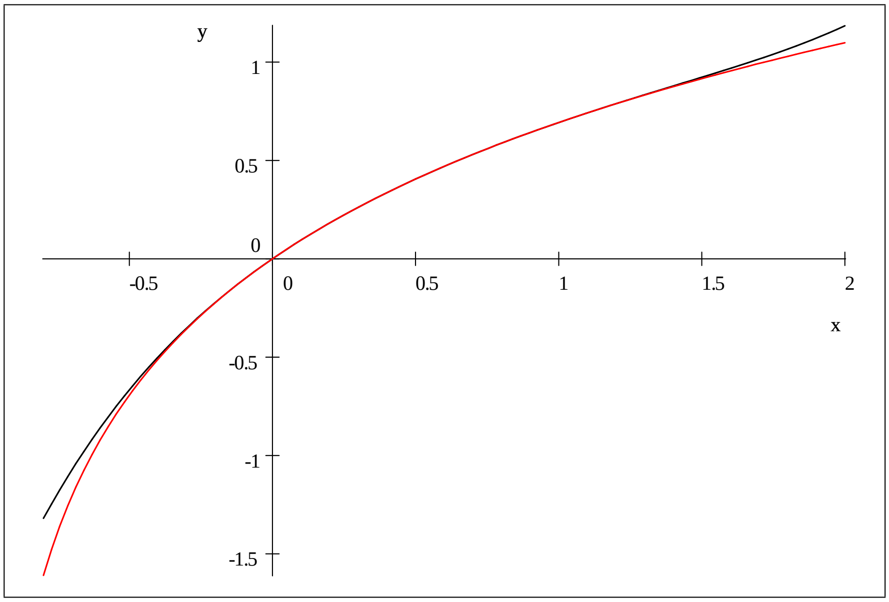
Example The lagrange polynomial $P_5(x)$ which interpolates the function $f(x) = \ln(1+x)$ at the nodes $0, 0.2, 0.4, 0.6, 0.8, 1$ on the interval $[0,1]$ is given by

$$\begin{aligned} P_5(x) &= \ln(1) L_{5,0} + \ln(1.2) L_{5,1} + \ln(1.4) L_{5,2} + \ln(1.6) L_{5,3} + \ln(1.8) L_{5,4} + \ln(2.0) L_{5,5} \\ &= \ln(1.2)\left(130.21x^5 - 364.58x^4 + 369.79x^3 - 160.42x^2 + 25.0x\right) \\ &\quad + \ln(1.4)\left(-260.42x^5 + 677.08x^4 - 614.58x^3 + 222.92x^2 - 25.0x\right) \\ &\quad + \ln(1.6)\left(260.42x^5 - 625.0x^4 + 510.42x^3 - 162.5x^2 + 16.667x\right) \\ &\quad + \ln(1.8)\left(-130.21x^5 + 286.46x^4 - 213.54x^3 + 63.542x^2 - 6.25x\right) \\ &\quad + \ln(2.0)\left(26.042x^5 - 52.083x^4 + 36.458x^3 - 10.417x^2 + x\right) \\ &= 0.9991x - 0.4891x^2 + 0.2825x^3 - 0.1290x^4 + .0296x^5 \end{aligned}$$

The polynomial $P_5(x)$ and the function $\ln(1+x)$ over the interval $[0,1]$ are shown on the same graph below.



If we extend the graphs to outside the interval $[0,1]$ the functions start to depart and the approximation is bad outside the nodes interval.



The polynomial $P_n(x)$ defined by equation (3) is called the Lgrange interpolating polynomial. The next theorem shows that the Lgrange interpolating polynomial is unique.

Theorem 2 *The Lgrange interpolating polynomial defined by equation (3) is unique.*

Proof. Suppose $Q_n(x)$ is a polynomial of degree at most n which interpolates f at the nodes $x_0, x_1, \ldots, x_n$. Then, we have

$$\begin{aligned} P_n(x_j) &= f(x_j), \\ Q_n(x_j) &= f(x_j), \ j = 0, 1, \ldots, n. \end{aligned}$$

Let $R(x) = P_n(x) - Q_n(x)$. Then $R(x)$ is a polynomial of degree at most n. Furthermore,

$$R(x_j) = P_n(x_j) - Q_n(x_j) = 0, \ j = 0, 1, \ldots, n.$$

By the fundamental theorem of algebra, a polynomial of degree at most n cannot have more than n roots unless it is identically zero. Since $R(x)$ has $n+1$ roots, we conclude that $R(x) \equiv 0$. Therefore, $P_n(x) - Q_n(x) \equiv 0$ and $P_n(x) \equiv Q_n(x)$.The Error Formula for Lagrange Interpolation ■

If the function f has $n+1$ continuous derivatives on $[a, b]$, we can prove the following error formula which closely resembles the error formula for the Taylor polynomials.

Theorem 3 *Suppose $f \in C^{n+1}[a, b]$ and P_n is the Lagrange interpolating polynomial at the nodes $x_0, x_1, \ldots, x_n$. Suppose also that $x \in [a, b]$ is such that $x_m < x < x_{m+1}$ for some m, where $0 \leq m < n$. Then there is a number $\xi \in (x_0, x_n)$ such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

Proof. Define the function $L(t)$ by

$$L(t) = \prod_{j=0}^{n} \frac{(t - x_j)}{(x - x_j)}$$

and the function $g(t)$ by

$$g(t) = f(t) - P_n(t) - [f(x) - P_n(x)] L(t)$$

Notice that $L(t)$ is a polynomial of degree $n+1$, $g \in C^{n+1}[a,b]$,

$$g(x_k) = f(x_k) - P_n(x_k) - [f(x) - P_n(x)] L(x_k) = 0, \;\; k = 0, 1, \ldots, n$$

and

$$\begin{aligned} g(x) &= f(x) - P_n(x) - [f(x) - P_n(x)] L(x) \\ &= f(x) - P_n(x) - [f(x) - P_n(x)] = 0. \end{aligned}$$

Applying Roll's theorem to the function g and the intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_m, x], [x, x_{m+1}], \ldots, [x_{n-1}$ we get that there exist $\xi_0^1 \in (x_0, x_1), \xi_1^1 \in (x_1, x_2), \ldots, \xi_m^1 \in (x_m, x), \xi_{m+1}^1 \in (x, x_{m+1}), \xi_n^1 \in [x_{n-1}, x_n]$ such that

$$g'\left(\xi_k^1\right) = 0, \;\; k = 0, 1, \ldots, n.$$

Observe that we have $x_0 < \xi_0^1 < \xi_1^1 < \ldots < \xi_m^1 < \xi_{m+1}^1 < \ldots < \xi_n^1 < x_n$. Applying Roll's Theorem again to the function g' and the intervals $\left[\xi_0^1, \xi_1^1\right], \left[\xi_1^1, \xi_2^1\right], \ldots, \left[\xi_{n-1}^1, \xi_n^1\right]$, we get the there exist $\xi_0^2 \in \left(\xi_0^1, \xi_1^1\right), \xi_1^2 \in \left(\xi_1^1, \xi_2^1\right), \ldots, \xi_{n-1}^2 \in \left(\xi_{n-1}^1, \xi_n^1\right)$ such that

$$g''\left(\xi_k^2\right) = 0, \;\; k = 0, 1, \ldots, n-1.$$

Continuing in this manner we ge that there exists a $\xi = \xi_0^{n+1} \in (\xi_0^n, \xi_1^n) \subset (x_0, x_n)$ such that

$$g^{(n+1)}(\xi) = 0.$$

Now

$$g^{(n+1)}(t) = f^{(n+1)}(t) - [f(x) - P_n(x)] (n+1)! \prod_{j=0}^{n} \frac{1}{(x - x_j)}.$$

Substituting $t = \xi$ and simplifying we get

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

■

The error formula

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)(x - x_1) \ldots (x - x_n)$$

can be used to give a crude error estimate as follows. For $x_m < x < x_{m+1}$, the product $|(x - x_0)(x - x_1)\ldots(x - x_n)|$ can be estimated by

$$|(x - x_0)(x - x_1)\ldots(x - x_n)| \le |(x - x_m)(x - x_{m+1})|(b - a)^{n-1}.$$

The term $|(x - x_m)(x - x_{m+1})| = (x - x_m)(x_{m+1} - x)$ has a maximum value (found using calculus) of

$$\frac{(x_{m+1} - x_m)^2}{4}$$

and this maximum occures when

$$x = \frac{x_{m+1} + x_m}{2}.$$

Therefore, we have

$$|(x - x_0)(x - x_1)\ldots(x - x_n)| \le \frac{(x_{m+1} - x_m)^2}{4}(x_n - x_0)^{n-1}$$

and

$$|E_n(x)| \le \frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}\frac{(x_{m+1} - x_m)^2}{4}(x_n - x_0)^{n-1}.$$

Example Let us estimate the error of interpolating the function $f(x) = \ln(1 + x)$ by the Lagrange polynomial $P_5(x)$ at the nodes $0, 0.2, 0.4, 0.6, 0.8, 1$.

$$\begin{aligned} |E_5(x)| &\le \frac{\left|f^{(6)}(\xi)\right|}{6!}\frac{(0.2)^2}{4}(1 - 0)^4 \\ &= \frac{1}{6!}\frac{6!}{(1+\xi)^6} \times .01 \le 0.01 \end{aligned}$$

This means that we should expect at least two decimal place accuracy in this approximation. The actual error (computed by taking $\max(\text{abs}(P_5(x) - \ln(x)))$ over the interval $[0, 1]$) is arround 10^{-5} so we actually have 5 decimal place accuracy.