Bézier Curves

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Bésier curves are widely used in CAD/CAM operations. They are the basis of the entire Adobe lllustrator, Macromedia Freehand and Photographer software products. Bésier curves curves are constructed using a special set of polynomials called the *Bernstein Polynomials*. We begin by introducing the Bernstein polynomials and exploring their elementary properties.

1 Bernstein Polynomials

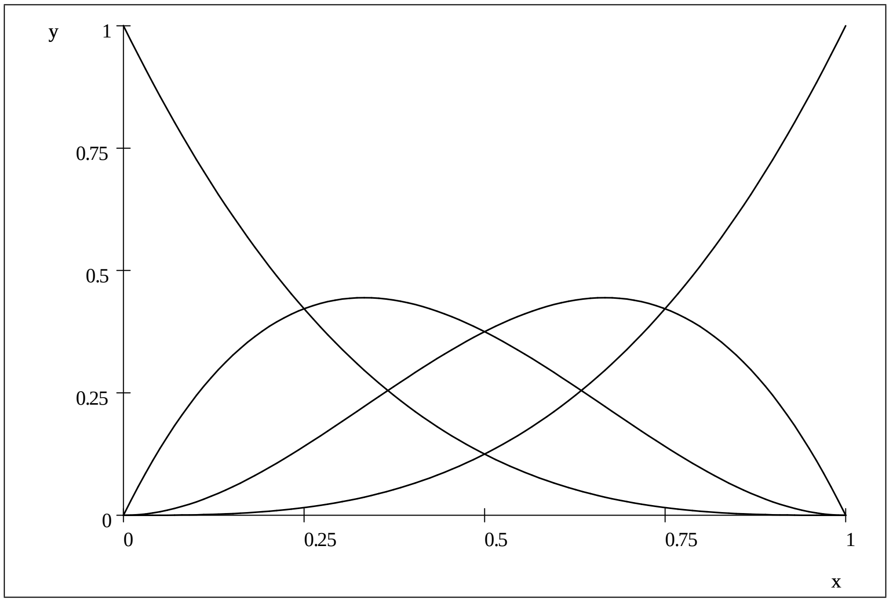
The Bernstein polynomials are defined by

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \; k = 0, 1, \ldots, n.$$

For example

$$\begin{aligned} B_{0,0} &= 1 \\ B_{0,1} &= 1-t, \; B_{1,1} = t \\ B_{0,2} &= (1-t)^2, \; B_{1,2} = 2t(1-t), B_{2,2} = t^2 \\ B_{0,3} &= (1-t)^3, \; B_{1,3} = 3t(1-t)^2, B_{2,3} = 3t^2(1-t), B_{3,3} = t^3 \end{aligned}$$

the Bernstein polynomials of degree 3 are shown below



1.1 Properties of Bernstein Polynomials

Following are some properties of Bernstein polynomials.

1. Boundary values

$$\begin{aligned} B_{0,n}(0) &= 1, B_{k,n}(0) = 0,\ 0 < k \leq n = B_{n,n}(1), \\ B_{n,n}(1) &= 1, B_{k,n}(1) = 0,\ 0 \leq k < n = B_{n,n}(1). \end{aligned}$$

2. Recurrence Relation

$$B_{k,n}(t) = (1-t) B_{k,n-1}(t) + t B_{k-1,n-1}(t).$$

   To see this, note first that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

   as can be verified by direct computation. Therefore,

$$\begin{aligned} \binom{n}{k} t^k (1-t)^{n-k} &= \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k-1} t^k (1-t)^{n-k} \\ &= (1-t) \binom{n-1}{k} t^k (1-t)^{n-1-k} + t \binom{n-1}{k-1} t^{k-1} (1-t)^{n-1-(k-1)} \\ &= (1-t) B_{k,n-1}(t) + t B_{k-1,n-1}(t). \end{aligned}$$

3. Nonnegativity

$$B_{k,n}(t) \geq 0 \ \forall t \in [0,1].$$

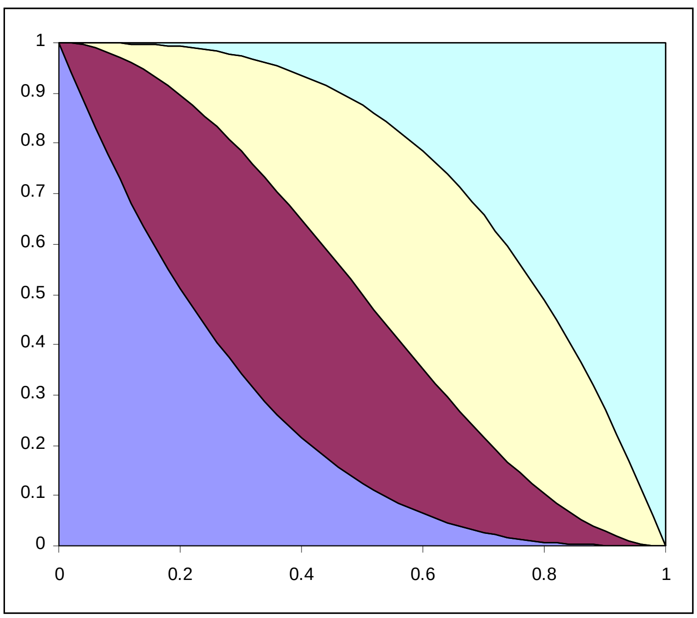
4. Partition of Unity

$$\sum_{k=0}^{n} B_{k,n}(t) = 1.$$

This is because

$$\begin{aligned} 1 &= 1^n = (1 - t + t)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \\ &= \sum_{k=0}^{n} B_{k,n}(t). \end{aligned}$$

The following figure shows the Bernstein polynomials of order 3 drawn on top of one another to verify that the sum of these polynomials at each t is one.



5. Derivatives

The Bernstein polynomials have derivatives of all orders and

$$\frac{d}{dt} B_{k,n}(t) = n \left(B_{k-1,n-1}(t) - B_{k,n-1}(t) \right)$$

6. Bases

The Bernstein polynomials of order n form a basis of all polynomials of degree less than or equal to n.

2 The Bézier Curves

A pair $(x, y) \in \mathbb{R}^2$ can be treated as a point in $\mathbb{R}^2$ or as a vector in In order to define the Bézier curves we should keep both points of view in mind.

Definition 1 *Let* $P_k = (x_k, y_k)\,,\ k = 0, 1, \ldots, n$ *be a set of points in* $\mathbb{R}^2$.*The Bézier curve of order* n *and controls* $\{P_k\}_{k=0}^{n}$ *is defined as follows*

$$\begin{aligned} P(t) &= \sum_{k=0}^{n} P_k B_{k,n}(t) \\ &= (x(t), y(t)) \end{aligned}$$

where

$$\begin{aligned} x(t) &= \sum_{k=0}^{n} x_k B_{k,n}(t), \\ y(t) &= \sum_{k=0}^{n} y_k B_{k,n}(t). \end{aligned}$$

For example, the Bézier curve of order 3 and controls $(2, 2), (1, 1.5), (3.5, 0)$ and $(4, 1)$ is

$$P(t) = (2, 2) B_{0,3}(t) + (1, 1.5) B_{1,3}(t) + (3.5, 0) B_{2,3}(t) + (4, 1) B_{3,3}(t).$$

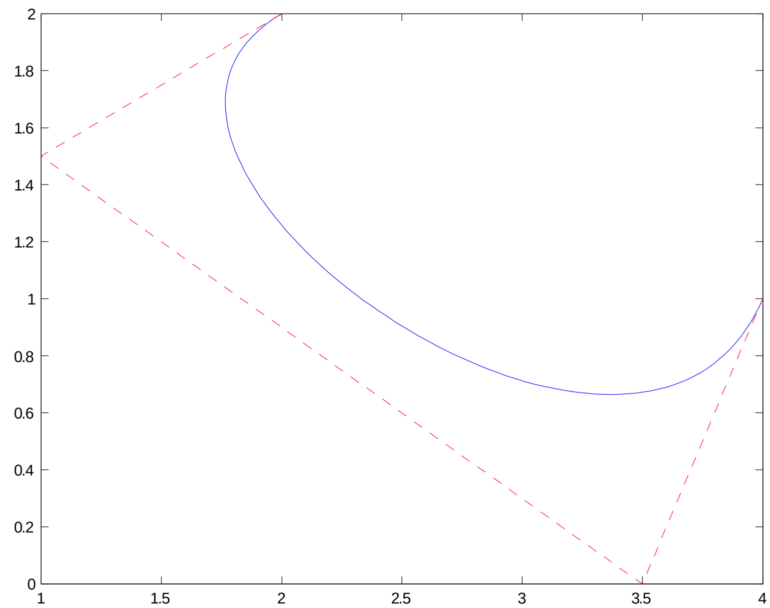
Therefore,

$$\begin{aligned} x(t) &= 2B_{0,3}(t) + 1B_{1,3}(t) + 3.5B_{2,3}(t) + 4B_{3,3}(t) \\ &= 2B_{0,3}(t) + B_{1,3}(t) + 3.5B_{2,3}(t) + 4B_{3,3}(t) \\ &= 2 - 3t + 10.5t^2 - 5.5t^3, \\ y(t) &= 2B_{0,3}(t) + 1.5B_{1,3}(t) + 0B_{2,3}(t) + 1B_{3,3}(t) \\ &= 2B_{0,3}(t) + 1.5B_{1,3}(t) + B_{3,3}(t) \\ &= 2 - 1.5t - 3t^2 + 3.5t^3. \end{aligned}$$

Thus

$$P(t) = \left(2 - 3t + 10.5t^2 - 5.5t^3, 2 - 1.5t - 3t^2 + 3.5t^3\right).$$

The following figure shows this Bézier curve together with its controls.



This curve is drawn as the set of points $(x(t), y(t)),\ t \in [0,1]$.

2.1 Properties of Bézier Curves

The following are elementary properties of Bézier curves.

1. Boundary values

$$P(0) = P_0, P(1) = P_n.$$

   This is because (using property 1 of Bernstein polynomials)

$$\begin{aligned} P(0) &= \sum_{k=0}^{n} P_k B_{k,n}(0) \\ &= P_0 B_{0,n}(0) = P_0, \\ P(1) &= \sum_{k=0}^{n} P_k B_{k,n}(1) \\ &= P_n B_{n,n}(1) = P_n. \end{aligned}$$

2. Boundary derivatives

$$\begin{aligned} P'(0) &= n(P_1 - P_0), \\ P'(1) &= n(P_n - P_{n-1}). \end{aligned}$$

   To see this we use property 5 of Bernstein polynomials to get (with $B_{j,m} = 0$ for

$j < 0$ or $j > m$)

$$\begin{aligned} P'(0) &= \sum_{k=0}^{n} P_k B'_{k,n}(0) \\ &= \sum_{k=0}^{n} P_k n \left(B_{k-1,n-1}(0) - B_{k,n-1}(0)\right) \\ &= n \sum_{k=1}^{n-1} P_k B_{k-1,n-1}(0) - n \sum_{k=0}^{n-1} P_k B_{k,n-1}(0) \\ &= n\left(P_1 - P_0\right) \end{aligned}$$

and similarly for $P'(1)$.

3. $P(t)$ has derivatives of all orders on the interval $[0,1]$.

4. Convexity

   A set $C \subset \mathbb{R}^2$ is called convex if given two points $P, Q \in C$, the line segment joining P and Q is entirely contained in C. That is, if the following condition is satisfied

   $$P, Q \in C, \;\; s \in [0,1] \Rightarrow (sP + (1 - sQ)) \in C.$$

   Given a set of points $\{P_k\}_{k=0}^{n}$ in $\mathbb{R}^2$ and a set of "weights" $\{m_k\}_{k=0}^{n}$ such that $m_k \geq 0$ and $\sum_{k=0}^{n} m_k = 1$, the linear combination

   $$\sum_{k=0}^{n} m_k P_k$$

   is called a convex combination of the points $\{P_k\}_{k=0}^{n}$. The smallest convex set containing $\{P_k\}_{k=0}^{n}$ is called the convex hull of $\{P_k\}_{k=0}^{n}$. It is known that the convex hull of the points $\{P_k\}_{k=0}^{n}$ equals precisely the set of convex combinations of $\{P_k\}_{k=0}^{n}$. Suppose now that we have a Bézier curve of order n and controls $\{P_k\}_{k=0}^{n}$.Since, for each $t \in [0,1]$, by properties 3 and 4 of Bernstein polynomials,

   $$P(t) = \sum_{k=0}^{n} P_k B_{k,n}(t)$$

   is a convex combination of $\{P_k\}_{k=0}^{n}$, we conclude that $P(t)$ lies in the convex hull of $\{P_k\}_{k=0}^{n}$. In other words, the entire Bézier curve lies entirely in the convex hull of $\{P_k\}_{k=0}^{n}$.

2.2 Composite Bézier Curves

In practice, curves are produced using a sequence of Bézier curves sharing common endpoints. The sequence of Bézier curves produced this way is called a composite Bézier curve.

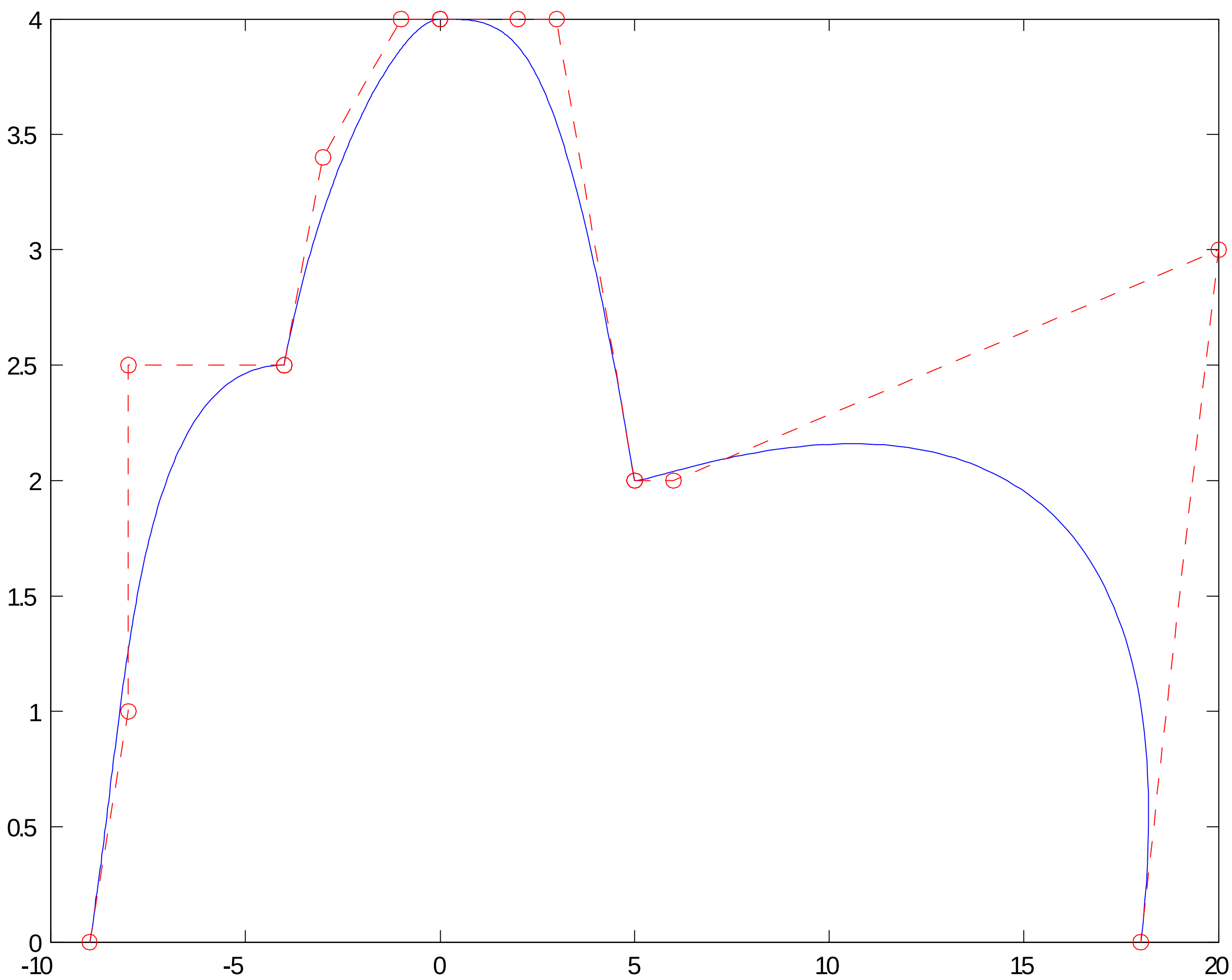
Example Find the composite Bézier curve for the four sets of control points

$$\{(-9,0),(-8,1),(-8,2.5),(-4,2.5)\}, \qquad \{(-4,2.5),(-3,3.5),(-1,4),(0,4)\},$$
$$\{(0,4),(2,4),(3,4),(5,2)\}, \qquad \{(5,2),(6,2),(20,3),(18,0)\}.$$

The composite Bézier curve consists of 4 pieces as follows

$$\begin{aligned}
P_1(t) &= (-9,0)B_{0,3}(t)+(-8,1)B_{1,3}(t)+(-8,2.5)B_{2,3}(t)+(-4,2.5)B_{3,3}(t), \\
P_2(t) &= (-4,2.5)B_{0,3}(t)+(-3,3.5)B_{1,3}(t)+(-1,4)B_{2,3}(t)+(0,4)B_{3,3}(t), \\
P_3(t) &= (0,4)B_{0,3}(t)+(2,4)B_{1,3}(t)+(3,4)B_{2,3}(t)+(5,2)B_{3,3}(t), \\
P_4(t) &= (5,2)B_{0,3}(t)+(6,2)B_{1,3}(t)+(20,3)B_{2,3}(t)+(18,0)B_{3,3}(t).
\end{aligned}$$

The following figure gives the composite Bézier curve for the above points



As this example shows, the pieces of the compostie Bézier curve form corners as they meet. If smooth transition from one piece to the next is required (say between the pieces

$P_i(t)$ and $P_{i+1}(t)$), then we must have

$$P_i'(1) = P_{i+1}'(0).$$

If $P_i(t)$ is of order n and has controls $\{P_k\}_{k=0}^n$ and $P_{i+1}(t)$ is of order m and has controls $\{Q_j\}_{j=0}^m$, then we must have

$$n(P_n - P_{n-1}) = m(Q_1 - Q_0).$$

or (since $P_n = Q_0$)

$$n(P_n - P_{n-1}) = m(Q_1 - P_n)$$

This means that the vector $(P_n - P_{n-1})$ is parallel to the vector $(Q_1 - P_n)$. Another way of putting this is to say that the points P_{n-1}, P_n, Q_1 must be on the same line. To illustrate, consider the Bézier curves $P(t)$ and $Q(t)$ of order 3 and controls

$$\{(0,3), (1,5), (2,1), (3,3)\} \qquad \text{and} \qquad \{(3,3), (4,5), (5,1), (6,3)\},$$

respectively. The controls $(2,1), (3,3), (4,5)$ are collinear and, therefore, we get the smooth composite curve shown below.