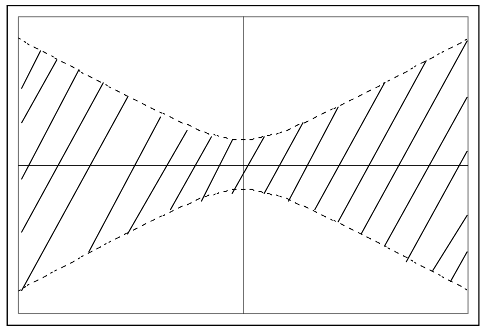
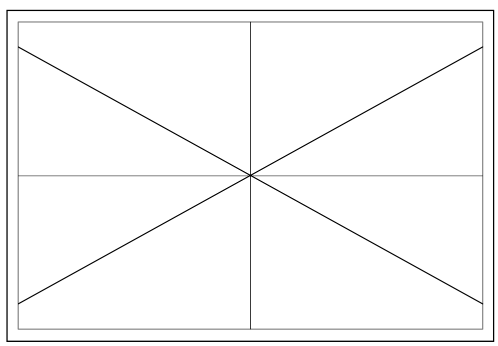
**Solutions of the third midterm**

1. (a) The domain consists of points $(x, y)$ such that $x^2 + y^2 - 1 > 0$. The sketch is shown below.

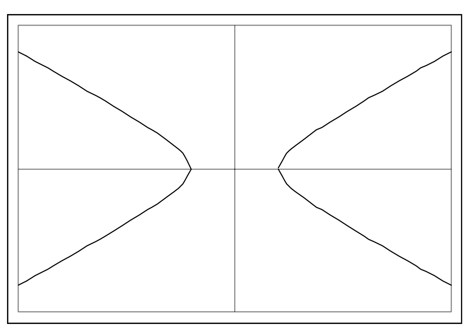


(b) $F(g(t), h(t)) = F\left(\sqrt{t}, t^3 - 1\right) = \ln\left(t - \left(t^3 - 1\right)^2 + 1\right) = \ln\left(t - t^6 + 2t^3\right)$.

(c) $\ln\left(x^2 - y^2 + 1\right) = 0 \Longrightarrow x^2 - y^2 + 1 = 1 \Longrightarrow x^2 - y^2 = 0$. The level curve is shown below.



$\ln\left(x^2 - y^2 + 1\right) = \ln 2 \Longrightarrow x^2 - y^2 + 1 = 2 \Longrightarrow x^2 - y^2 = 1$. The level curve is shown below.



2. (a) $\nabla f = \left\langle \frac{y^2}{x}, 2y\ln x \right\rangle$. $\nabla f(P) = \langle 1, 0\rangle$. Unit vector in the direction of **a**,

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

$D_{\mathbf{u}} f(P) = \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{5}}$

(b) Unit vector is $\langle 1, 0\rangle$, rate of change is $\|\nabla f(P)\| = 1$

(c) Unit vector is $\langle -1, 0\rangle$, rate of change is $-\|\nabla f(P)\| = -1$

(d) $f$ has zero rate of change in the direction perpindicular to $\nabla f(P)$, i.e., $\langle 0, 1\rangle$ or $\langle 0, -1\rangle$

3. (a) Approach through $y = x$ :

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2} = \lim_{x\to 0} \frac{x^4}{2x^6 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^4 + 1} = 0.$$

Approach through $y = x^3$ :

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2} = \lim_{x\to 0} \frac{x^6}{2x^6 + x^6} = \frac{1}{3}.$$

Since the two limits are different, the limit does not exist.

(b) Put $r = x^2 + y^2 - 5$. Then

$$\lim_{(x,y)\to(1,2)} \frac{\sin\left(5 - x^2 - y^2\right)}{2x^2 + 2y^2 - 10} = \lim_{r\to 0} \frac{\sin(-r)}{2r} = -\frac{1}{2}.$$

4. (a) Substitute from the parametric equations of the line into the equation of the surface to get

$$2t + 7 = (-1 + t)^2 + (2 + t)^2.$$

Solve this quadratic equation in $t$ to obtain $t = \pm 1$. Substitute back in the parametric equations to get the two points of intersection $(0, 3, 9)$ and $(-2, 1, 5)$.

(b) Set $f(x, y, z) = x^2 + y^2 - z$. $\nabla f = \langle 2x, 2y, -1\rangle$. At the point $(0, 3, 9)$, $\nabla f = \langle 0, 6, -1\rangle$.
Equation of the tangent plane: $6(y - 3) - (z - 9) = 0$. Equations of normal line:$x = 0$, $y = 3 + 6t$, $z = 9 - t$. At the point $(-2, 1, 5)$, $\nabla f = \langle -4, 2, -1\rangle$.Equation of the tangent plane: $-4(x + 2) + 2(y - 1) - (z - 5) = 0$. Equations of normal line:$x = -2 - 4t$, $y = 1 + 2t$, $z = 5 - t$.

5. (a) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \left(3x^2y^2z^4\right)(2t) + \left(2x^3yz^4\right) + \left(4x^3y^2z^3\right)\left(8t^3\right).$

at $t=1,\ x=1,\ y=3,\ z=2$. substituting in the above expression we get

$$\frac{dw}{dt}=3264$$

(b) $f(x,y,z)=ye^x-5\sin 3z-3z$.

$$\begin{aligned} z_x &= -\frac{f_x}{f_z}=\frac{ye^x}{15\cos 3z+3}, \\ z_y &= -\frac{f_y}{f_z}=\frac{e^x}{15\cos 3z+3}. \end{aligned}$$

6. (a)

$$\begin{aligned} L(x,y) &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= \left(x_0^2+y_0^2\right)+2x_0(x-x_0)+2y_0(y-y_0) \\ &= 2y-2x-2. \end{aligned}$$

Comparing the coefficients of $x,\ y$ on both sides, we get $x_0=-1,\ y_0=1$. i.e., $P=(-1,1)$.

(b) $\triangle f\approx df=f_x(P)\triangle x+f_y(P)\triangle y$. $f_x(P)=\frac{1}{3}x^{-2/3}y^{1/2}\big|_{(x,y)=(8,9)}=\frac{1}{4}$. $f_x(P)=\frac{1}{2}x^{1/3}y^{-1/2}\big|_{(x,y)=(8,9)}=\frac{1}{3}$. $\triangle x=-0.22,\ \triangle y=0.03$. Hence, $df=-0.045$.