Exam # 2

1. (a) **(4 points)** Find the equation of the tangent plane and the parametric equations of the normal line to the surface $xy + \ln(y/z) = 8$ at the point $(4, 2, 2)$.

   (b) **(4 points)** Calculate $\frac{\partial x}{\partial w}$ and $\frac{\partial z}{\partial w}$ for $xe^w + we^z = ze^x$.

   Solution:

$$\begin{aligned} \nabla f &= \left\langle y, x + \frac{1}{y}, -\frac{1}{z} \right\rangle. \\ \nabla f(4,2,2) &= \left\langle 2, \frac{9}{2}, -\frac{1}{2} \right\rangle. \end{aligned}$$

Equation of the tangent plane:

$$2(x-4) + \frac{9}{2}(y-2) - \frac{1}{2}(z-2) = 0.$$

Equations of the normal line

$$x = 4 + 2t,\ y = 2 + \frac{9}{2}t,\ z = 2 - \frac{1}{2}t.$$

(b):

$$\begin{aligned} F(x,w,z) &= xe^w + we^z - ze^x \\ x_w &= -\frac{F_w}{F_x} = -\frac{xe^w + e^z}{e^w - ze^x}. \\ z_w &= -\frac{F_w}{F_z} = -\frac{xe^w + e^z}{we^z - e^x}. \end{aligned}$$

2. (a) **(4 points)** A right circular cone had radius 120 in. and hight 140 in. if the error in measuring the radius is 1.8 in and the error in measuring the hight is -2.5 in. use differentials to estimate the error in calculating the volume of the cone. (The volume of a right circular cone is $V = \frac{\pi}{3}r^2h$.)

   (b) **(3 points)** Find all points on the line $x = 1 + t, y = 2 - 3t, z = 4 + 2t$ that are at the same distance from the two planes $x - 2y + 3z = 1,\ 2x + 3y + z = 2$.

   Solution: (a)

$$\begin{aligned} dV &= V_r dr + V_h dh \\ &= \frac{2\pi}{3} rhdr + \frac{\pi}{3} r^2 dh \\ &= \frac{\pi}{3}(2 * 120 * 140 * 1.8 - 120 * 120 * 2.5) \\ &= 8160\pi \text{ in}^3. \end{aligned}$$

(b) Distance from first plane:

$$\frac{|(1+t) - 2(2-3t) + 3(4+2t) - 1|}{\sqrt{14}} = \frac{|13t + 8|}{\sqrt{14}}.$$

Distance form second plane:

$$\frac{|2(1+t)+3(2-3t)+(4+2t)-2|}{\sqrt{14}}=\frac{|10-5t|}{\sqrt{14}}$$

Equality of the two distances means

$$\begin{aligned} 13t+8 &= \pm(10-5t). \\ t &= \frac{1}{9},\ t=-\frac{9}{4}. \end{aligned}$$

The points are

$$P_1=\left(\frac{10}{9},\frac{5}{3},\frac{38}{9}\right),\ P_2=\left(-\frac{5}{4},\frac{35}{4},-\frac{1}{9}\right).$$

3. (a) **3 points)** Find the minimum rate of change of the function $f(x,y,z)=xy\sin(xz)$ at the point $\left(1,-1,\frac{\pi}{3}\right)$ and the direction in which it occures.

(b) **(4 points)** Find all directions $\mathbf{u}$ in which the function $f(x,y)=x^2+2y$ has slope 1 at the point $(1,0)$.

Solution: (a):

$$\begin{aligned} \nabla f &= \langle y\sin xz+xy\cos xz, x\sin xz, x^2y\cos xz\rangle \\ \nabla f\left(1,-1,\frac{\pi}{3}\right) &= \left\langle -\frac{\sqrt{3}}{2}-\frac{\pi}{6},\frac{\sqrt{3}}{2},-\frac{1}{2}\right\rangle \\ &= \langle -1.3896, 0.86603, -0.5\rangle. \end{aligned}$$

Minimum rate of change $=-\left\|\nabla f\left(1,-1,\frac{\pi}{3}\right)\right\|=\sqrt{(-1.3896)^2+(0.86603)^2+(-0.5)^2}=$ 1.712 and it occures in the direction

$$\begin{aligned} \mathbf{u} &= -\frac{\nabla f\left(1,-1,\frac{\pi}{3}\right)}{\left\|\nabla f\left(1,-1,\frac{\pi}{3}\right)\right\|}=\left\langle\frac{-1.3896}{1.712},\frac{0.86603}{1.712},\frac{-0.5}{1.712}\right\rangle \\ &= \langle -0.81168, 0.50586, -0.29206\rangle. \end{aligned}$$

(b):

$$\begin{aligned} \nabla f &= \langle 2x, 2\rangle \\ \nabla f(1,0) &= \langle 2,2\rangle. \end{aligned}$$

If we write the unit vector $\mathbf{u}=\langle u_1,u_2\rangle$ then $u_1^2+u_2^2=1$ and

$$\begin{aligned} D_uf &= 2u_1+2u_2=1 \\ u_1+u_2 &= \frac{1}{2}, u_1^2+u_2^2=1 \\ u_1 &= \frac{1}{2}-u_2 \\ \left(\frac{1}{2}-u_2\right)^2+u_2^2 &= 1. \end{aligned}$$

Solving the last equation using the quadratic formula we obtain

$$u_2 = \frac{1 \pm \sqrt{7}}{4}, u_1 = \frac{1 \mp \sqrt{7}}{4}.$$

Therefore,

$$\mathbf{u} = \left\langle \frac{1+\sqrt{7}}{4}, \frac{1-\sqrt{7}}{4} \right\rangle, \text{ or } \mathbf{u} = \left\langle \frac{1-\sqrt{7}}{4}, \frac{1+\sqrt{7}}{4} \right\rangle.$$

4. (a) **(4 points)** Find the equation of the plane that passes through the three points $(1,0,0), (0,2,-2), (-5,2,1)$.

   (b) **(4 points)** Identify and sketch the surface .

   Solution: (a): We use the three points to form two vectors

$$\mathbf{u}_1 = \langle -1, 2, -2 \rangle, \ \mathbf{u}_2 = \langle -6, 2, 1 \rangle.$$

The normal to the plane is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -2 \\ -6 & 2 & 1 \end{vmatrix} = 6\mathbf{i} + 13\mathbf{j} + 10\mathbf{k}.$$

Equation of the plane is

$$6(x-1) + 13y + 10z = 0.$$

(b): The surface is an elliptic paraboloid with axis along the z-axis.